On the Use of Independent Base-Stock Policies in Assemble-to-Order Inventory Systems with Nonidentical Lead Times

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We consider Independent Base-Stock (IBS) policies in Assemble-to-Order (ATO) inventory systems with a general bill of materials and deterministic nonidentical lead times. The main drawback of IBS policies in this setting is that they keep the inventory position of each component constant without considering the availability of other components with longer lead times. We show that the cost of this deficiency is capped by an upper bound that is proportional to the ratio of the maximum difference of lead times to the shortest lead time when the latter is below a threshold, and to the square root of this ratio otherwise. Our result indicates that IBS policies can remain effective when differences of lead times are dominated by their lengths, and can be asymptotically optimal as lead times increase while their differences grow at slower rates.

Key words: base-stock policies, nonidentical lead times, assemble-to-order, stochastic program.

1. Introduction

Independent base-stock (IBS) policies are widely discussed in the literature on Assemble-to-Order (ATO) inventory systems, where the objective is to minimize the long-run average expected total inventory cost (e.g., see [6] for a review of many papers that follow this approach). For systems with identical lead times, IBS policies with properly-chosen base-stock levels are asymptotically optimal in the long-lead time or high-volume asymptotic regimes ([4]). They can even be exactly optimal in several special cases ([2]). However, as can be shown by sample path comparisons, for systems with nonidentical lead times, IBS policies can be dominated by other policies that adjust inventory positions of some components according to the availability of other components with longer lead times. As has been proven by studies on single-product ATO systems ([5]), or more general single-product assembly systems ([8]), the optimal policy in these cases is a “balanced
echelon” base-stock policy, which does not keep inventory positions constant for all components except the ones with the longest lead time.

Despite this deficiency, because of difficulties of defining, optimizing, and implementing alternative policies, IBS policies remain an attractive option for managing ATO inventory systems. In fact, as is pointed out in the literature review of [7], many studies and practical implementations follow an ISS policy, which is a special form of IBS policy that ignores simultaneous stock-out of components when setting base stock levels. Hence assessing the cost of applying IBS policies in systems with nonidentical lead times is an interesting and important problem, which we address in this paper. We consider systems with a general bill of materials and deterministic but different lead times. Our analysis reveals that in many cases, under a suitable IBS policy, the long-run average expected total inventory cost can stay close to its minimum.

We start from an existing lower bound on the cost objective in [3], which is obtained by solving a $K + 1$ stage stochastic program (SP), where $K$ is the number of distinct lead times. We collapse the SP into a two-stage SP and show that its objective value is also a lower bound. The first-stage solution of the SP is used to set base stock levels for our IBS policy. Observe that this approach differs from ISS policies by not ignoring simultaneous stock-out. This policy and the demand arrival process jointly determine the probability distribution of component balance (demand - supply). For each realization of the balance, the second-stage of the SP produces an allocation solution that minimizes the total inventory cost. Therefore, inventory management in an ATO system is perfectly optimal if at each point of time, the replenishment policy can induce the same probability distribution of component balance as that in the SP, and for any given realization of component balance, the allocation policy can replicate the outcome of the second stage of the SP.

In general, differences in lead times make it impossible to match the component balance between ATO systems under an IBS policy and their corresponding SPs. However, we prove that the cost of this mismatch, as a percentage of the SP lower bound, is bounded above by a quantity that is proportional to the ratio of the maximum difference of lead times to the shortest lead time when the latter is below a threshold, and to the square root of that ratio otherwise.

As implications, when component allocation is as perfect as that in the SP, which is the case with single-product ATO systems and some special multi-product systems ([3]), the IBS policy prescribed in this paper results in an inventory cost that is close to its lower bound when the variations of lead times are trivial in comparison with their lengths. For other cases where the perfect allocation is not attainable, it has been shown ([4]) that there exist asymptotically optimal allocation policies in the long lead time regime. Therefore a combined use of these allocation policies and our IBS replenishment policy will also be asymptotically optimal as the lead times grow while their differences grow at a slower rate.
Our results apply to many situations in practice. For instance, when products are assembled from components supplied by oversea vendors, delay on the common route of import-export control and inter-continental transportation can be far longer than differences in supplier-specific delays. For these cases, our results suggest that the loss of optimality by following a properly-designed IBS policy is small.

The rest of the paper is organized as follows: Section 2 formulates the model. Section 3 develops the aforementioned two-stage SP. Section 4 develops an IBS policy and its performance bounds. Section 5 discusses implications of latter results on the overall optimality of inventory management.

2. Problem Formulation

We develop our analysis for the continuous-review formulation. Our results also extend to the periodic-review model by analogous arguments.

We consider ATO systems with \( m \) products and \( n \) components. The bill of materials (BOM) is given by a non-negative integer matrix \( A \), where \( a_{ji} \) denotes the usage of component \( j \) by product \( i \) \((1 \leq j \leq n, 1 \leq i \leq m)\). Suppose that there are \( K \) distinct lead times with \( L_K > L_{K-1} \cdots > L_1 > 0 \). We use \( k_j \) \((k_j = 1, \ldots, K)\) to refer to the index of the lead time of component \( j \), so that the lead time of component \( j \) is \( L_{k_j} \), \( 1 \leq j \leq n \). Let \( n_0 = 0 \) and \( n_k \) be the number of components with lead time \( L_k \) or shorter \((1 \leq k \leq K)\), so \( n_K = n \). Let components be indexed according to the ascending order of their lead times, so \( \{n_{k-1} + 1, \ldots, n_k\} \) is the index set of components with lead time \( L_k \( (1 \leq k \leq K)\). Let \( A^k \) be a sub-matrix of \( A \) that corresponds to the usage of components with lead time \( L_k \( (1 \leq k \leq K)\). Then

\[
A = \begin{pmatrix}
A^1 \\
A^2 \\
\vdots \\
A^K
\end{pmatrix}.
\]

We will also use \( A_j \) (bold and with a subscript) to refer to row \( j \) of \( A \), which defines the usage of component \( j \) \((1 \leq j \leq n)\) by all products.

Demands arrive according to a compound Poisson process. The number of orders during the time interval \([0, t]\) is denoted by \( \Lambda(t) \( t \geq 0\) \) and

\[
\Lambda = E[\Lambda(1)]
\]

is the order arrival rate. A generic element of this sequence is denoted by \( S = (S_1, S_2, \ldots, S_m) \), where \( S_i \) be the order size of product \( i \), \( 1 \leq i \leq m \). We assume that \( S_i \) has a finite second moment \((1 \leq i \leq m)\). (In Section 5, for one of our results we assume that \( S_i \) has a finite moment of order \( 2 + \delta \) where \( \delta \) is a positive value that can be arbitrarily small.) The total quantity of product demands during time \([0, t]\) is denoted by

\[
D(t) = (D_1(t), \cdots, D_m(t))', \quad t \geq 0,
\]
with $E[\mathcal{D}(1)] = \mu < \infty$. For the convenience of our analysis, define

$$\mathbf{D}(t_1, t_2) = \mathcal{D}(t_2) - \mathcal{D}(t_1), \quad 0 \leq t_1 \leq t_2,$$

as the amount of demand that arrives during the interval $(t_1, t_2]$. Let

$$\mathbf{D}^k(t) = \mathbf{D}(t - L_k, t - L_{k-1}), \quad 1 \leq k \leq K, \quad t \geq L_K,$$

and

$$\tilde{\mathbf{D}}^k(t) = \mathbf{D}(t - L_K, t - L_k), \quad 0 \leq k \leq K, \quad t \geq L_K,$$

where $L_0 = 0$ (note that $\tilde{\mathbf{D}}^0(t) = \mathbf{D}(t - L_K, t)$). Since demand processes are stationary, we define random vectors $\mathbf{D}^k$ and $\tilde{\mathbf{D}}^k$ such that

$$\mathbf{D}^k \overset{d}{=} \mathbf{D}^k(t) \ (1 \leq k \leq K) \text{ and } \tilde{\mathbf{D}}^k \overset{d}{=} \mathbf{D}^k(t) \ (0 \leq k \leq K), \quad t \geq L_K.$$

Let $\mathcal{R}_j(t)$ be the amount of component $j$ ordered between time $-L_{k_j}$ and time $t$ for $t \geq -L_{k_j}$ and define

$$\mathcal{R}(t) = (\mathcal{R}_1(t), \ldots, \mathcal{R}_n(t))', \quad t \geq 0.$$

Denote the total amounts of demands served during $[0, t]$ by

$$\mathcal{Z}(t) = (\mathcal{Z}_1(t), \ldots, \mathcal{Z}_m(t))', \quad t \geq 0.$$

Similarly, for $1 \leq k \leq K$, denote

$$\mathbf{R}^k(t) = \mathcal{R}(t) - \mathcal{R}(t - L_k) \quad \text{and} \quad \mathbf{Z}^k(t) = \mathcal{Z}(t) - \mathcal{Z}(t - L_k), \quad t \geq 0.$$

We consider a backlog model in which the backlog levels at time $t$ are denoted by $\mathbf{B}(t) = (B_1(t), \ldots, B_m(t))'$ and the per-unit backlog costs are given by $\mathbf{b} = (b_1, \ldots, b_m)'$. Let $\mathbf{I}^k(t) = (I_{n_k-1+1}(t), \ldots, I_{n_k}(t))'$ be the inventory levels of components with lead time $L_k$, and $\mathbf{h}^k = (h_{n_k-1+1}, \ldots, h_{n_k})'$ be the corresponding unit inventory holding costs ($1 \leq k \leq K$). Let

$$\mathbf{I}(t) = \begin{pmatrix} \mathbf{I}^1(t) \\ \mathbf{I}^2(t) \\ \vdots \\ \mathbf{I}^K(t) \end{pmatrix}, \quad t \geq 0, \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^K \end{pmatrix}.$$

Changes of backlog and inventory levels are governed by

$$\mathbf{B}(t) = \mathbf{B}(t - L_k) + \mathbf{D}(t - L_k, t) - \mathbf{Z}^k(t), \quad 1 \leq k \leq K, \quad t \geq t_K,$$

and

$$\mathbf{I}^k(t) = \mathbf{I}^k(t - L_k) + \mathbf{R}^k(t - L_k) + A^k \mathbf{Z}^k(t), \quad 1 \leq k \leq K, \quad t \geq t_K.$$  \hspace{1cm} (1)

The objective is to minimize the long-run average expected inventory cost

$$C \equiv \limsup_{T \to \infty} \frac{1}{T} \int_{L_K}^{T + L_K} C(t) dt,$$

where

$$C(t) = \mathbf{b} \cdot E[\mathbf{B}(t)] + \mathbf{h} \cdot E[\mathbf{I}(t)] = \mathbf{b} \cdot E[\mathbf{B}(t)] + \sum_{k=1}^{K} h^k \cdot E[\mathbf{I}^k(t)].$$  \hspace{1cm} (2)
3. Stochastic Programs

Under a feasible inventory policy, backlog and inventory levels are nonnegative at all times. Replenishment and allocation policies are only based on information available when decisions are made. For the family of policies that satisfy the above conditions, Reiman and Wang ([3]) show that

\[ C \equiv b \cdot \sum_{k=1}^{K} E[D^k] + \inf_{\alpha \in R_n} \{ b \cdot \alpha + \Phi^K(\alpha) \} \]  

is a lower bound on the inventory cost (2), where \( \Phi^K(\alpha) \) is the optimal solution of the following multi-stage stochastic program (SP)

\[
\Phi^K(\alpha) = \inf_{y^K \geq 0} \{ h^K \cdot y^K + E[\Phi^{K-1}(y^K, \alpha + D^K)] \},
\]

\[
\Phi^k(y^{k+1}, \ldots, y^K, x) = \inf_{y^k \geq 0} \{ h^k \cdot y^k + E[\Phi^{k-1}(y^k, \ldots, y^K, x + D^k)] \}, \quad k = 1, \ldots, K - 1,
\]

\[
\Phi^0(y^1, \ldots, y^K, x) = -\max_{z \geq 0} \{ c \cdot z | z \leq x, Az \leq (y^1, \ldots, y^K) \},
\]

where \( c = b + A^T h \).

To put this formulation into perspective, the SP imitates a single-minded minimization of the expected inventory cost at a given time \( t \) in an ATO system. Determining \( y^k \) at stage \( k \) corresponds to choosing inventory levels of components with lead time \( L_k \) at time \( t - L_k \) (1 \( \leq k \leq K \)). Determining \( z \) in the last stage LP corresponds to optimizing component allocation to serve demands for different products. In addition to these decisions, backlogs at \( t - L_K \), denoted by \( \alpha \), also affect the inventory cost at time \( t \). A larger \( \alpha \) always helps (in the weak sense): at worst, they can be removed by increasing inventory levels by \( A\alpha \) to keep the inventory cost intact, but the additional backlog provides more flexibility, which might be used to further reduce the inventory cost. To set a lower bound, we may need some components of \( \alpha \) to go to infinity. In any case,

\[
\inf_{\alpha \in R_n} \{ b \cdot \alpha + \Phi^K(\alpha) \} = \lim_{\alpha \to \infty} \{ b \cdot \alpha + \Phi^K(\alpha) \}.
\]

The SP in (4) has \( K + 1 \) stages, where \( K \) is the number of distinct lead times. We relax this SP into one that has only two stages, for which the optimal solution can be imitated in ATO systems by an IBS policy. The optimal solution of the latter makes the lower bound even lower (in the weak sense).

Specifically, define

\[ C^* = b \cdot E[D^1] + \lim_{\alpha \to \infty} \{ b \cdot \alpha + \Phi(\alpha) \} \]  

where \( \Phi(\alpha) \) is the optimal solution of the following two-stage SP

\[
\Phi(\alpha) = \inf_{y \geq 0} \{ h \cdot y + E[\Phi^0(y, \alpha + D^1)] \}
\]

where \( \Phi^0(y, \alpha + D^1) = -\max_{z \geq 0} \{ c \cdot z | z \leq \alpha + D^1, Az \leq y \} \).
Theorem 1
\[ C^* \leq C \] (7)

Proof  To bridge the gap between the multi-stage SP in (4) and the two-stage SP in (6), we first define below an auxiliary SP, which, while multi-stage in formulation, is two-stage in essence. Let
\[ \psi^K(\alpha) = \inf_{y^K \geq 0} \{ h^K \cdot y^K + \psi^{K-1}(y^K, \alpha + E[D^K]) \}, \]
\[ \psi^k(y^{k+1}, ..., y^K, x) = \inf_{y^k \geq 0} \{ h^k \cdot y^k + \psi^{k-1}(y^k, ..., y^K, x + E[D^k]) \}, \quad k = K - 1, ..., 2, \]
\[ \psi^1(y^2, ..., y^K, x) = \Phi^1(y^2, ..., y^K, x) = \inf_{y^1 \geq 0} \{ h^1 \cdot y^1 + E[\psi^0(y^1, ..., y^K, x + D^1)] \}, \]
\[ \psi^0(y^1, ..., y^K, x) = \Phi^0(y^1, ..., y^K, x) = -\max_{z \geq 0} \{ c \cdot z | x, Az \leq (y^1, ..., y^K)' \}. \]

Applying the above to expand \( \psi^K(\alpha) \) progressively,
\[ \psi^K(\alpha) = \inf_{y^1, ..., y^K} \left\{ \sum_{k=1}^K h^k \cdot y^k + E[\psi^0(y^1, ..., y^K, \alpha + E[D^K] + ..., D^2] + D^1] \right\} \]
\[ = \Phi(\alpha + E[D^K + ... + D^2]). \]

It follows that
\[ \lim_{\alpha \to \infty} (b \cdot \alpha + \Phi(\alpha)) = b \cdot \sum_{k=2}^K E[D^k] + \lim_{\alpha \to \infty} (b \cdot \alpha + \psi^K(\alpha)). \]

Since
\[ C^* = b \cdot E[D^1] + \lim_{\alpha \to \infty} (b \cdot \alpha + \Phi(\alpha)) \quad \text{and} \quad C = b \cdot \sum_{k=1}^K E[D^k] + \lim_{\alpha \to \infty} (b \cdot \alpha + \Phi^K(\alpha)), \]
we can prove the theorem by showing that
\[ \Phi^K(\alpha) \geq \psi^K(\alpha) \quad \text{for all} \quad \alpha \in \mathbb{R}^m_+, \] (8)

which we do next.

As an LP property (cf. [1], p. 213), \( \Phi^k(y^{k+1}, ..., y^K, x) \) is convex in \( x \) when \( k = 0 \), and since convexity is preserved after taking the infimum and the expectation, it is also convex in \( x \) when \( k = 1, ..., K \).

By definition, when \( k = 0 \) or 1,
\[ \Phi^k(y^{k+1}, ..., y^K, x) \geq \psi^k(y^{k+1}, ..., y^K, x). \]

To carry out an induction, assume it is true for some \( k \geq 1 \), then
\[ \Phi^{k+1}(y^{k+2}, ..., y^K, x) = \inf_{y^{k+1} \geq 0} \{ h^{k+1} \cdot y^{k+1} + E[\Phi^k(y^{k+1}, ..., y^K, x + D^{k+1})] \} \]
\[ \geq \inf_{y^{k+1} \geq 0} \{ h^{k+1} \cdot y^{k+1} + \Phi^k(y^{k+1}, ..., y^K, x + E[D^{k+1}]) \} \]
\[ \geq \inf_{y^{k+1} \geq 0} \{ h^{k+1} \cdot y^{k+1} + \psi^k(y^{k+1}, ..., y^K, x + E[D^{k+1}]) \} \]
\[ = \psi^{k+1}(y^{k+2}, ..., y^K, x), \]
where the first inequality follows from Jensen’s Inequality and the second one from the induction hypothesis. We complete our proof at \( k = K - 1 \). ■

As \( \alpha \) in (5) approaches infinity, the optimal \( y \) of SP (6) becomes unbounded and is thus difficult to imitate by a feasible policy in ATO systems. The lemma below originates from [4] and shows that the same \( C^* \) can be obtained from an alternative SP with a finite optimal solution.

**Lemma 1** Let

\[
\Phi = \min_{y \in \mathbb{R}^n} \{ h \cdot y + E[\tilde{\Phi}(y, D^1)] \}
\]

where \( \tilde{\Phi}(y, x) = -\max_{z \in \mathbb{R}^m} \{ c \cdot z | z \leq x, Az \leq y \} \).

Then

\[
\tilde{\Phi} = \lim_{\alpha \to \infty} (b \cdot \alpha + \Phi(\alpha)),
\]

and thus

\[
C'^* = b \cdot E[D^1] + \tilde{\Phi}.
\]

Moreover, there exists a finite constant \( M \) such that for any optimal solution of (9), \( y^* = (y^*_1, ..., y^*_n)' \),

\[ |y^*_j| \leq M, \quad 1 \leq j \leq n. \]

**Proof** Refer to Theorem 2 in [4] and apply (5). ■

4. **IBS Policy and Performance Bound**

In this section, we define a base-stock replenishment policy based on the optimal solution of (9), develop a performance bound, and discuss situations when the policy is close to optimal.

Let \( y^* \) be the optimal solution of (9) and \( y^{k*} \) be a part of this solution that corresponds to components with lead time \( L_k \) \((1 \leq k \leq K)\). Under our IBS policy, the base-stock levels are given by

\[
Y^k = y^{k*} + (L_k - L_1)A^k \mu, \quad 1 \leq k \leq K.
\]

Observe that for any \( t \geq 0 \),

\[
(L_k - L_1)A^k \mu = A^k E[D(t - L_k, t - L_1)], \quad 1 \leq k \leq K.
\]

Intuitively, (9) prescribes \( y^{k*} \) \((1 \leq k \leq K)\) as the amounts of components for serving demands occurring over a period of length \( L_1 \). For components with lead time \( L_1 \), we use these quantities as base-stock levels. For a component \( j \) with lead time \( L_{k_j} > L_1 \), we start from \( y^*_j \) and add the mean demand for the component over a period of length \( L_{k_j} - L_1 \) \((n_1 + 1 \leq j \leq n)\) to set base-stock levels.
To evaluate the performance of this policy, we compare relative quantities in the ATO system with their counterparts in the lower bound SP (9). Define
\[ Q = A \mathbf{D}^1 - \mathbf{y}^* \] (13)
as the component balance (i.e., demand - supply) in the SP (9). Component \( j \) has shortage when \( Q_j > 0 \) \((1 \leq j \leq n)\), in which case backlogs become inevitable. At given balance \( Q \), the optimal backlog levels are given by
\[ \min_{B \geq 0} \{c \cdot B | A B \geq Q \}, \] (14)
which is equivalent to the second-stage LP of (9). Let \( z^* \) be an optimal solution of the latter LP. Then
\[ B^* = D^1 - z^*, \]
is an optimal solution of (14). The reverse is also true.

In ATO systems, component balances are given by
\[ Q^k(t) = A^k \mathbf{B}(t) - \mathbf{I}^k(t), \quad 1 \leq k \leq K, \quad t \geq 0. \]

Observe that by the definition of our IBS policy, for \( 1 \leq k \leq K \),
\[ \mathbf{Y}^k = \mathbf{y}^{k*} + A^k \mu(L_k - L_1) \quad \text{and} \quad \mathbf{Y}^k = \mathbf{I}^k(t - L_k) + \mathbf{R}^k(t - L_k) - A^k \mathbf{B}(t - L_k), \quad t \geq 0. \]

Applying the above and (1) to eliminate \( \mathbf{B}(t) \) and \( \mathbf{I}^k(t) \) \((1 \leq k \leq K)\), balances of components with lead time \( k \) \((1 \leq k \leq K)\) can also be expressed as
\[ Q^k(t) = A^k \mathbf{D}^1(t) - \mathbf{y}^{k*} + A^k \sum_{l=2}^{k} \left( \mathbf{D}^l(t) - \mathbf{E}[\mathbf{D}^l(t)] \right), \quad t \geq 0. \] (15)

The lemma below shows that the difference of the expected inventory cost in an ATO system from its SP lower bound is completely reflected by the difference between the amounts of unserved demands in these two cases.

**Lemma 2** Let \( C(t) \) be the expected inventory cost under our IBS policy at time \( t \) \((t \geq L_K)\) and \( C^* \) be the lower bound given in (11). Then
\[ C(t) - C^* = c \cdot (\mathbf{E}[\mathbf{B}(t)] - \mathbf{E}[\mathbf{B}^*]). \] (16)

**Proof** By definition, given base-stock levels \( \mathbf{Y}^k \),
\[ \mathbf{I}^k(t - L_k) + \mathbf{R}^k(t - L_k) = \mathbf{Y}^k + A^k \mathbf{B}(t - L_k), \quad 1 \leq k \leq K. \]
Apply the above and (1), the expected inventory cost at time $t$ is

$$C(t) = \sum_{k=1}^{K} h^k \cdot E[I^k(t)] + b \cdot E[B(t)] = c \cdot E[B(t)] + \sum_{k=1}^{K} h^k \cdot (Y^k - E[A D(t - L_k, t)]) .$$

Let $z^*$ be the optimal solution of $\tilde{\Phi}^0(y^*, D^1)$ in (10). Define $B^* = D^1 - z^*$. Then $B^*$ is the optimal solution of (14) and

$$C^* = b \cdot E[D^1] + \sum_{k=1}^{K} h^k \cdot y^k - c \cdot E[z^*] = \sum_{k=1}^{K} h^k \cdot y^k + c \cdot E[B^*] - \sum_{k=1}^{K} h^k \cdot E[A D^1].$$

The lemma follows because by (12),

$$Y^k - E[A D(t - L_k, t)] = y^k - E[A D^1], \quad 1 \leq k \leq K. \quad \blacksquare$$

Observe that $E[B(t)]$ and $E[B^*]$ can differ for two reasons. First referring to (13) and (15), our IBS policy may induce component balances that differ in distribution from those in the SP. Second, for given component balance, the outcome of optimizing backlog levels by (14) may not be replicated by a feasible allocation policy for controlling ATO systems. To separate these two influences, we define $B^*(t)$ as the optimal solution of

$$\min_{B \geq 0} \left\{ c \cdot B | A^k B \geq Q^k(t), 1 \leq k \leq K \right\}, \quad (17)$$

and decompose the relative difference of the inventory cost from its lower bound into the following two parts

$$\frac{C(t) - C^*}{C^*} = \frac{c \cdot (E[B^*(t)] - E[B^*])}{C^*} + \frac{c \cdot (E[B(t)] - E[B^*(t)])}{C^*}. \quad (18)$$

The first part

$$\frac{c \cdot (E[B^*(t)] - E[B^*])}{C^*} \quad (19)$$

assumes an optimized allocation solution for both the SP and the inventory system, and thus isolates the effect of different component balances under our base-stock policy and the SP solution. The second part

$$\frac{c \cdot (E[B(t)] - E[B^*(t)])}{C^*} \quad (20)$$

measures the difference of the backlog cost from the optimized allocation outcome (which may not be achievable), given the same balance process induced by our IBS policy. Since the purpose of this paper is to discuss base-stock policies, we focus on (19), and only discuss briefly the design of an allocation policy for minimizing (20) in the next section.

To proceed with our analysis, define

$$D^k = D^2 + \cdots + D^k.$$  

Notice that for $j = 1, \ldots, n$, $A_j \cdot D^k_j$ represents the amount of component $j$ needed to serve all demands arrived during a period of length $L_{k_j} - L_1$. 
Lemma 3 There exists a finite constant $\kappa_1$, which depends only on values of $c$ and $A$, such that

$$c \cdot (E[B^*(t)] - E[B^*]) \leq \kappa_1 E \left[ \sum_{j=1}^{n} |A_j \cdot D^k_j - E[A_j \cdot D^k_j]| \right].$$  \hspace{1cm} (21)

**Proof** Since $D^1(t) = d$, we can compare $B^*(t)$ and $B^*$ on matching sample paths ($D^1(t) = D^1$). From (14) and (17), $B^*$ and $B^*(t)$ are optimal solutions of the same LP with different RHS values. Following standard LP sensitivity analysis, there exists a finite constant $\kappa_1$ such that

$$c \cdot (B^*(t) - B^*) \leq \kappa_1 \max_{1 \leq j \leq n} |Q_j(t) - Q_j|.$$

Here $\kappa_1$ can be defined as the maximum absolute value of all elements of the vectors in the feasible set of the dual LP,

$$\{x : A^T x \leq c; x \geq 0\},$$

which is finite. It follows that

$$c \cdot (E[B^*(t)] - E[B^*]) \leq \kappa_1 E \left[ \max_{1 \leq j \leq n} |Q_j(t) - Q_j| \right]$$

$$= \kappa_1 E \left[ \max_{1 \leq j \leq n} \sum_{k=2}^{k_j} A_j \cdot D^k_j - E \left[ \sum_{k=2}^{k_j} A_j \cdot D^k_j \right] \right]$$

$$= \kappa_1 E \left[ \max_{1 \leq j \leq n} |A_j \cdot D^j_j - E \left[ A_j \cdot D^j_j \right]| \right]$$

$$\leq \kappa_1 E \sum_{j=1}^{n} \left| A_j \cdot D^j_j - E \left[ A_j \cdot D^j_j \right] \right|,$$

where the first equality comes from (13) and (15). $\blacksquare$

Lemma 4 There exists a finite constant $\kappa_2$, which depends only on values of $b$, $h$, and $A$, such that

$$C^* \geq \kappa_2 \sum_{j=1}^{n} E \left[ |A_j \cdot D^1 - E[A_j \cdot D^1]| \right].$$  \hspace{1cm} (22)

**Proof** From Lemma 1 and (9),

$$C^* = \sum_{i=1}^{m} b_i E[D_i^1 - z_i^*] + \sum_{j=1}^{n} h_j E[y_j^* - A_j \cdot z^*].$$

Let

$$\tilde{b} = \min_{1 \leq i \leq m} \left\{ \frac{b_i}{\sum_{j=1}^{n} a_{ji}} \right\}.$$

Then

$$\sum_{i=1}^{m} b_i (D_i^1 - z_i^*) \geq \tilde{b} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ji} (D_i^1 - z_i^*) = \tilde{b} \sum_{j=1}^{n} (A_j \cdot D^1 - A_j \cdot z^*).$$
Since $z^* \leq D^1$ and $A_j \cdot z^* \leq y_j^*$, $(1 \leq j \leq n)$,
\[
C^* \geq \sum_{j=1}^{n} \left\{ b_j E[(A_j \cdot D^1 - y_j^*)^+] + h_j E[(y_j^* - A_j \cdot D^1)^+] \right\}.
\]

Define $F_j(x)$ as the CDF of $A_j \cdot D^1$ and $\bar{F}_j(x) = 1 - F_j(x)$, $(1 \leq j \leq n)$. Let
\[
q_j = \frac{\bar{F}_j(E[A_j \cdot D^1])}{F_j(E[A_j \cdot D^1])}, \quad 1 \leq j \leq n.
\]
Let
\[
b_j = \tilde{b} \wedge (h_j q_j) \quad \text{and} \quad h_j = h_j \wedge (\tilde{b}/q_j), \quad 1 \leq j \leq n.
\]
Then
\[
b_j \leq \tilde{b}, \quad h_j \leq h_j, \quad \text{and} \quad \bar{F}_j(E[A_j \cdot D^1]) = \frac{b_j}{b_j + h_j}, \quad 1 \leq j \leq n.
\]
Hence $E[A_j \cdot D^1]$ is the optimal solution of the Newsvendor model
\[
\min_x \left\{ b_j E[(A_j \cdot D^1 - x)^+] + h_j E[(x - A_j \cdot D^1)^+] \right\}, \quad 1 \leq j \leq n,
\]
and it follows that
\[
C^* \geq \sum_{j=1}^{n} \left\{ b_j E[(A_j \cdot D^1 - y_j^*)^+] + h_j E[(y_j^* - A_j \cdot D^1)^+] \right\}
\geq \sum_{j=1}^{n} \left\{ b_j E[(A_j \cdot D^1 - E[A_j \cdot D^1])^+] + h_j E[(E[A_j \cdot D^1] - A_j \cdot D^1)^+] \right\}
\geq \sum_{j=1}^{n} (b_j \wedge h_j) E[||A_j \cdot D^1 - E[A_j \cdot D^1||].
\]

The lemma follows by defining $\kappa_2 \equiv \min_{1 \leq j \leq n} (b_j \wedge h_j)$. ■

Using the above two lemmas, the next theorem establishes an upper bound on the difference between product backlogs in the ATO system and its corresponding SP (19).

**Theorem 2** There exist a threshold $\bar{L}$ and finite constants $\beta_1, \beta_2 > 0$ that do not depend on $t$ such that
\[
\frac{c \cdot (E[B^*(t)] - E[B^*])}{C^*} \leq \beta_1 \left( \frac{L_K}{L_1} - 1 \right),
\]
if $L_1 \leq \bar{L}$. Otherwise
\[
\frac{c \cdot (E[B^*(t)] - E[B^*])}{C^*} \leq \beta_2 \sqrt{\frac{L_K}{L_1} - 1}.
\]

**Proof** Let $D$ be a random vector such that $D \overset{d}{=} D(1)$. Define
\[
X_j^0 = A_j \cdot D
\]
as the demand for component \( j \) \((1 \leq j \leq n)\) within a unit interval, which has a finite mean \( E[X_j^0] \) and a finite standard deviation \( \sigma_{X_j^0} \). Define

\[
X_j = A_j \cdot D^k, \quad 1 \leq j \leq n.
\]

Since \( X_j \) is positive and compound Poisson, its mean absolute deviation (MAD) satisfies

\[
E[|X_j - E[X_j]|] \leq 2E[X_j] = 2(L_{kj} - L_1)E[X_j^0], \quad 1 \leq j \leq n.
\]  

(26)

The MAD is also bounded by the standard deviation, i.e.,

\[
E[|X_j - E[X_j]|] \leq \sigma_{X_j} = \sigma_{X_j^0} \sqrt{L_{kj} - L_1}, \quad 1 \leq j \leq n.
\]  

(27)

Based on (21), we apply (26) and (27) to bound \( c \cdot (E[B^*(t)] - E[B^*]) \). Define

\[
W_j \equiv A_j \cdot D^1, \quad 1 \leq j \leq n.
\]

Based on our assumption of \( D^1 \), \( W_j \) is compound Poisson. Denote

\[
\Pr(\Lambda(L_1) = l) = p_l
\]

as the probability that the number of orders during \([0, L_1]\) is \( l \) \((l = 0, 1, \ldots)\) and

\[
\lambda_1 \equiv E[\Lambda(L_1)] = \bar{\lambda}L_1
\]

(recall that \( \bar{\lambda} \) is the order arrival rate). Recall that \( S = (S_1, \ldots, S_m) \) is a generic element of the order size sequence with \( S_i \) the order size for product \( i \), \( 1 \leq i \leq m \). The product order size \( S \) translates into a component order size of

\[
s_j = A_j \cdot S, \quad 1 \leq j \leq n.
\]

Let \( \bar{s}_j \) and \( \sigma_{s_j} \) denote, respectively, the mean and standard deviation of \( s_j \), \( 1 \leq j \leq n \). They are all finite, by our previous assumptions. It follows that

\[
E[W_j] = \lambda_1 s_j \quad \text{and} \quad \sigma_{W_j} = \sigma_{s_j} \sqrt{\lambda_1}, \quad 1 \leq j \leq n.
\]

Let

\[
\nu_j = (\sqrt{2} + 1) \frac{\sigma_{s_j} + 1}{\bar{s}_j}, \quad 1 \leq j \leq n.
\]

By the Central Limit Theorem, for a Poisson variable \( \Lambda \) with mean \( \lambda \), there exists some \( \bar{\lambda}_j \) such that for all \( \lambda \geq \bar{\lambda}_j \),

\[
\Pr(\Lambda \leq [\lambda - \nu_j \sqrt{\lambda}]) \geq \Pr \left( \frac{\Lambda - \lambda}{\sqrt{\lambda}} \leq -\nu_j - \frac{1}{\sqrt{\lambda}} \right) \geq 0.8 \Pr(-\nu_j), \quad 1 \leq j \leq n,
\]
where $\mathcal{P}(x)$ is the CDF of standard Normal distribution. Define

$$u_j = \bar{\lambda}_j \lor [v_j^2], \quad 1 \leq j \leq n,$$

which do not depend on $L_1$. Define $\bar{L} = \min_{1 \leq i \leq n} \{u_j/\lambda\}$. When $L_1 \leq \bar{L}$, $\lambda_1 = \lambda L_1 \leq u_j$ for all $j$. Therefore

$$\mathbb{E}[[W_j - \mathbb{E}[W_j]]] \geq p_0 \mathbb{E}[W_j] \geq e^{-u_j} \lambda_1 \bar{s}_j = e^{-u_j} \lambda L_1 \bar{s}_j, \quad 1 \leq j \leq n. \tag{28}$$

Using Lemma 3 and (26) to set an upper bound on the numerator below on the left-hand side and Lemma 4 and (28) to set a lower bound on the denominator, there exists a finite constant $\beta_1$ that does not depend on $t$ such that

$$\frac{c \cdot (\mathbb{E}[B^+(t)] - \mathbb{E}[B^+]]}{C^*} \leq \beta_1 \left( \frac{L_K}{L_1} - 1 \right), \tag{29}$$

proving (24).

When $L_1 > \bar{L}$, $\lambda_1 \geq u_j$ for some $j$. Fix such a component $j$, and let

$$\xi_j = (\sigma_{s_j} + 1).$$

Then for $l = 1, ..., \lfloor \lambda_1 - v_j \sqrt{\lambda_1} \rfloor$, $l \geq 0$ because $\sqrt{\lambda_1} \geq \sqrt{u_j} \geq v_j$, and

$$(\lambda_1 - l) \bar{s}_j - \xi_j \sqrt{\lambda_1} \geq (v_j \bar{s}_j - \xi_j) \sqrt{\lambda_1} = \sqrt{2} (\sigma_{s_j} + 1) \sqrt{\lambda_1}. \tag{30}$$

Let $s_j^i$ ($i = 1, ...$) be i.i.d copies of $s_j$. Then for all $l = 1, ..., \lfloor \lambda_1 - v_j \sqrt{\lambda_1} \rfloor$,

$$\Pr \left( \sum_{i=1}^{l} s_j^i > \mathbb{E}[W_j] - \xi_j \sqrt{\lambda_1} \right) = \Pr \left( \sum_{i=1}^{l} s_j^i - l \bar{s}_j > (\lambda_1 - l) \bar{s}_j - \xi_j \sqrt{\lambda_1} \right) \leq \frac{l \sigma_{s_j}^2}{[(\lambda_1 - l) \bar{s}_j - \xi_j \sqrt{\lambda_1}]^2} \leq \frac{1}{2}, \tag{31}$$

where the first inequality uses Chebyshev’s inequality, and the second one uses (30) along with $l < \lambda_1$. It follows that

$$\mathbb{E}[[W_j - \mathbb{E}[W_j]]] = \mathbb{E} \left[ \sum_{l=0}^{\lambda_1 - v_j \sqrt{\lambda_1}} \sum_{i=1}^{l} s_j^i - \lambda_1 \bar{s}_j \right] \geq \sum_{l=0}^{\lambda_1 - v_j \sqrt{\lambda_1}} p_l \mathbb{E} \left[ \sum_{i=1}^{l} s_j^i - \lambda_1 \bar{s}_j \right] \geq \xi_j \sqrt{\lambda_1} \sum_{l=0}^{\lambda_1 - v_j \sqrt{\lambda_1}} p_l \Pr \left( \sum_{i=1}^{l} s_j^i \leq \lambda_1 \bar{s}_j - \xi_j \sqrt{\lambda_1} \right) \geq \frac{\xi_j \sqrt{\lambda_1}}{2} \Pr \left( \Lambda(L_1) \leq \lambda_1 - v_j \sqrt{\lambda_1} \right) \geq 0.4 \xi_j \sqrt{\lambda_1} \mathcal{P}(-v_j),$$

where $\mathcal{P}(x)$ is the CDF of standard Normal distribution. Define

$$u_j = \bar{\lambda}_j \lor [v_j^2], \quad 1 \leq j \leq n,$$
where the penultimate inequality comes from (31) and the last inequality is due to $\lambda_1 \geq u_j$. Using the above and Lemma 4 to set a lower bound on $C^*$ and Lemma 3 and (27) to set an upper bound on the numerator in the left hand side below, there exists a finite constant $\beta_2 > 0$ that does not depend on $t$ such that

$$\frac{c \cdot (E[B^*(t)] - E[B^*])}{C^*} \leq \beta_2 \left( \sqrt{\frac{L_K}{L_1}} - 1 \right),$$

proving (25).

5. Implications

Consider an ATO system with lead times $L = (L_1, ..., L_K)$ where $L_1 < ... < L_K$. Denote the cost objective in (2) by $C^{(L)}$ and its lower bound in (11) by $C^{*(L)}$. Let $B^{(L)}(t)$ be the backlog levels at time $t$ and $B^{*(L)}(t)$ be the values defined in (17) ($t \geq 0$). The following corollary to Theorem 2 shows that if components can be allocated perfectly at each point of time, then the percentage difference of the average inventory cost under our IBS policy from its lower bound diminishes to zero as the ratio of the longest and the shortest lead times converges to unity.

**Corollary 1** If the allocation policy results in $c \cdot E[B^{(L)}(t)] = c \cdot E[B^{*(L)}(t)]$, $t \geq 0$, then under our IBS replenishment policy,

$$\lim_{L_K/L_1 \to 1} \frac{C^{(L)} - C^{*(L)}}{C^{*(L)}} = 0.$$

**Proof** Immediate from (18) and Theorem 2.

Perfect allocation is trivially attainable in one-product ATO systems, and also in multi-product systems with special structures and parameter values ([3]). While perfect allocation is often impossible in more general cases, there are still, albeit more restrictive, parameter regions in which our IBS policy satisfies above optimality criterion.

Reiman and Wang consider systems with an identical lead time $L$ for all components ([4]), and develop the following allocation principle: let $Q^{(L)}(t)$ be component balances at time $t$ ($t \geq 0$) (which they refer to as component shortage by considering any surplus inventory over existing demands as negative shortage). Set backlog targets at $B^{*(L)}(t)$, an optimal solution of (17) selected to be Lipschitz continuous in $Q^{(L)}(t)$. Any product with its backlog level currently below the target (i.e., $B_i^{(L)}(t) < B_i^{*(L)}(t)$, $1 \leq i \leq m$) is not served. All other products are served to clear as much excess backlogs as possible, subject to component availability. They prove that ([4]) under any allocation policy that satisfies this principle,

$$\lim_{L \to \infty} \sup_{t \geq L} \frac{c \cdot [E[B^{(L)}(t)] - E[B^{*(L)}(t)]]}{C^{*(L)}} = 0.$$

The following corollary extends asymptotic optimality of this allocation principle to our systems.
Corollary 2 If, for some $\delta > 0$, the demand order size $S_i$ ($1 \leq i \leq m$) has a finite moment of order $2 + \delta$ and $L_K/L_1 \to 1$ as $L_1 \to \infty$, then under our IBS policy and the above allocation principle,

$$\frac{C^{(L)} - C_\ast^{(L)}}{C_\ast^{(L)}} \to 0 \quad \text{as} \quad L_1 \to \infty.$$ 

Based on (18) and Theorem 2, we can prove the corollary by showing that

$$\lim_{L_1 \to \infty} \sup_{t \geq L_K} c \cdot \left( E[B^{(L)}(t)] - E[B_\ast^{(L)}(t)] \right) = 0,$$

which can be proven by a similar analysis as in Section 4 of [4]. To accommodate nonidentical lead times, we scale our systems by the shortest lead time $L_1$. The centered and scaled amount of demand arriving over a longer lead time $L_k$ ($2 \leq k \leq K$) satisfies

$$\frac{D_i(t - L_k, t) - L_k \mu_i}{\sqrt{L_1}} = \Delta \hat{D}_i^{(L_1)}(t) + \hat{D}_i^{(L_1)}(t),$$

where

$$\Delta \hat{D}_i^{(L_1)}(t) = \frac{D_i(t - L_k, t - L_k) - (L_k - L_1) \mu_i}{\sqrt{L_1}},$$

$$\hat{D}_i^{(L_1)}(t) = \frac{D_i(t - L_1, t) - L_1 \mu_i}{\sqrt{L_1}}, \quad 1 \leq i \leq m, \quad t \geq 0.$$ 

Applying the Central Limit Theorem, and because of stationarity of compound Poisson Processes (so that $E[\Delta \hat{D}_i^{(L_1)}(t)]$ does not depend on $t$) and the aforementioned finiteness of $2 + \delta$ moments, if $L_K/L_1 \to 1$ as $L_1 \to \infty$, then

$$\lim_{L_1 \to \infty} \sup_{t \geq L_K} E[\Delta \hat{D}_i^{(L_1)}(t)] = 0, \quad 1 \leq i \leq m.$$ 

This allows us to adopt the same proof of Theorem 4 in [4] to prove (32), with all additional terms arising from lead time differences eliminated by the above.

To conclude, while IBS policies are generally not optimal for ATO systems with nonidentical lead times, the policy developed in this paper is asymptotically optimal when the lead times grow but their differences do not grow as fast.

References