Pricing Bermudan Options in Lévy Models

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Abstract: This paper presents a Hilbert transform method for pricing Bermudan options in Lévy models. The corresponding optimal stopping problem is reduced to a backward induction in the Fourier space that involves taking Hilbert transforms of certain analytic functions or integrating such functions. The Hilbert transforms and integrals can be discretized using very simple schemes. The resulting discrete approximation can be efficiently implemented using the fast Fourier transform. The computational cost is linear in the number of monitoring times, and \( O(M \log(M)) \) in the number of points used to approximate the Hilbert transforms and integrals. The method is remarkably accurate. The pricing error decays exponentially in terms of the computational cost \( M \) for a wide class of Lévy process models. The early exercise boundary is obtained as a by-product. American options can be priced by increasing the number of monitoring times and using extrapolation when applicable.

1 Introduction

Option contracts are actively traded both on exchanges and in the over-the-counter market. It is well known that the commonly used Black-Scholes-Merton model ([3], [16]) understates the likelihood of extreme price movements in financial markets. One popular class of alternative models are Lévy process models ([17], [13], [1], [6], [15], [5]). They relax the restrictive assumptions of the Black-Scholes-Merton model by allowing jumps in the underlying asset price.

Most option contracts traded on exchanges and in the over-the-counter market are of American style and hence can be early exercised. It is of great interest to develop efficient methods for pricing options with early exercise features. In this paper, we focus on pricing Bermudan style options, which can be early exercised on a discrete set of monitoring times. American options can be priced by increasing the number of monitoring times and using extrapolation when applicable.

The valuation of Bermudan options corresponds to a discrete optimal stopping problem, which usually admits no analytical solution and must be solved numerically. The optimal stopping problem can be implemented using a backward induction, where at each monitoring time, one computes a conditional expectation representing the continuation value of the option and compare it with the option payoff representing the profit of immediate exercise. The main objective of the Bermudan option valuation problem is thus computing these conditional expectations. Methods based on monte carlo simulation, double exponential fast Gauss transform, Fourier-cosine series, lattice approximation, and numerical solution of partial integro-differential equations have been studied in [14], [4], [7], [12], and [9]. [11] proposes a Fourier space time-stepping method that uses the fact that the conditional expectation is essentially a convolution of the option value at a previous time step and the transition density of the Lévy process. Its Fourier transform is thus a product of the Fourier transform of the option value at the previous time step and the characteristic function of the Lévy process. The algorithm thus proceeds as follows: knowing the option value function at the previous time step (on a certain grid), one computes
its Fourier transform using a trapezoidal rule, multiplies the result by the characteristic function, and computes the Fourier inverse integral of the resulting product. The method is second order accurate due to the trapezoidal sum approximations of the Fourier and Fourier inverse integrals. As is also shown in [8], the pricing error of such an approach converges polynomially when Newton-Cotes type schemes are used.

In this paper, we show that, instead of switching between the Fourier space and the state space, one can conduct most of the computations in the Fourier space directly. Moreover, the simple trapezoidal rule in the Fourier space becomes remarkably accurate, with exponentially decaying errors, due to powerful approximation theory for analytic functions. This is in contrast to the second order accuracy of the trapezoidal scheme in the state space, as exhibited in [11].

Our method is based on the key observation that monitoring the early exercise boundary for a Bermudan option in the state space corresponds to taking a Hilbert transform in the Fourier space. More specifically, at each monitoring time, the option value is equal to the maximum of the option payoff and the continuation value. There exists a critical asset price level (the early exercise boundary) such that on one side of this level, the option value equals the payoff, and on the other side, the option value equals the conditional expectation. The option value function can thus be expressed as a summation of the payoff multiplied by an indicator function and the conditional expectation multiplied by another indicator function. This leads to a Hilbert transform representation for the Fourier transform of the option value function. Our method thus proceeds as follows: knowing the Fourier transform of the option value at the previous time step, one computes the Fourier transform of the option value at the current time step using the Hilbert transform representation. A final Fourier inversion leads to the option price at time zero. Our method thus involves a sequential evaluation of Hilbert transforms. The Hilbert transforms are discretized using very simple trapezoidal type schemes, yet with remarkable accuracy. The discretization error converges exponentially in terms of $1/h$, where $h$ is the step size used to approximate the Hilbert transforms. We truncate the resulting infinite sums with truncation level $M$. The discrete approximation then corresponds to a Toeplitz matrix vector multiplication, which can be implemented using the fast Fourier transform. As for the early exercise boundary, it solves the equation where the payoff equals the conditional expectation. We obtain a Fourier inverse representation for the conditional expectation, which can be discretized using the trapezoidal rule, again with exponentially decaying errors.

The early exercise boundary is then found using the Newton-Raphson method, which requires only a few iterations to achieve great accuracy. The computational cost of our method is $O(NM \log(M))$, where $N$ is the number of monitoring times, and $M$ is the truncation level. Moreover, for a wide class of Lévy processes, we present a convenient procedure for selecting the step size $h$ as a function of the truncation level $M$. The resulting pricing error then converges to zero exponentially in terms of $M$. This is very convenient in practice since we only need to worry about one control parameter, $M$, which represents the computational cost of the method. In the following, we describe the Hilbert transform formulation and present numerical results exhibiting the performance of the method.

## 2 Hilbert transform pricing of Bermudan options

### 2.1 Representation in the state space

We assume that the underlying asset price is governed by a geometric Lévy process under a given equivalent martingale measure $\mathbb{P}$:

$$S_t = S_0 e^{X_t}.$$ 

Here $X_t = \ln(S_t/S_0)$ is a Lévy process starting at the origin at time 0. $S_0$ is the initial asset price. Under the equivalent martingale measure, the discounted gain process is a martingale. That is, the following martingale condition is satisfied:

$$\mathbb{E}[S_t] = S_0 \mathbb{E}[e^{X_t}] = S_0 \phi_t(-i) = S_0 e^{(r-q)t}. \quad (2.1)$$

Here $r$ is the risk free interest rate, and $q$ is the continuous yield the underlying asset is paying.

In the following, we consider pricing Bermudan vanilla put options in geometric Lévy models. Suppose the current time is $t_0 = 0$. Denote the set of monitoring times for the Bermudan option by $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$. Here $t_N = T$ is the option maturity. To simplify notations and without loss of generality, we assume a constant monitoring interval $\Delta$. That is, $t_j = j\Delta \; j = 0, 1, \ldots, N$. A Bermudan option can be exercised at any time in $\mathcal{T}$. When exercised at time $t_n \in \mathcal{T}$, the payoff of a Bermudan put option is $G(S_{t_n})$, where $G(S) = (K - S)^+$. Here $K$ is the strike price of the option.

The valuation of a Bermudan option corresponds to solving a discrete optimal stopping problem. The value at time $0$ of the option is given by

$$V^0(S_0) = \sup_{\tau} \mathbb{E}[e^{-r\tau} G(S_\tau)],$$

where the sup is taken over all stopping times $\tau$ in $\mathcal{T}$. The Bermudan option can be exercised at any time in $\mathcal{T}$. When exercised at time $t_n \in \mathcal{T}$, the payoff of a Bermudan put option is $G(S_{t_n})$, where $G(S) = (K - S)^+$. Here $K$ is the strike price of the option.
where the supremum is taken over the set of all stopping times which take value in $T$. The above optimal stopping problem can be solved using a backward induction. We perform a change of variable $x = \ln(S/K)$ and consider, for any $0 < j < N$, there exists a critical value $-\infty < x^*_j < 0$ so that when $x \leq x^*_j$, early exercise is optimal and the maximum above equals the payoff function $g$. Otherwise, it is optimal to hold the option and the maximum above equals the discounted conditional expectation. $S^*_j = Ke^{r_j}$ is known as the early exercise boundary of the Bermudan option. Then we obtain the following expression for $f^j$:

$$f^j(x) = g(x) \cdot 1_{(-\infty,x^*_j]}(x) + e^{-r\Delta}E_{\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] \cdot 1_{(x^*_j,\infty)}(x).$$

We will implement the above backward induction in the Fourier space. To guarantee integrability so that the Fourier transform can be taken, we need to introduce an exponential dampening factor. For an appropriate $a$, we define $f^j_\alpha(x) = e^{a\Delta}f^j(x)$. Denote $g_\alpha(x) = e^{a\Delta}g(x)$. Suppose $f^j_\alpha \in L^1$. Then

$$e^{a\Delta}E_{\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] = \phi_\Delta(ia)E_{\Delta,x}[f^{j+1}_\alpha(X_{(j+1)\Delta})].$$

Here $\phi_\Delta(\cdot)$ is the characteristic function of $Xt$, $E_{\Delta,x}$ denotes the expectation conditional on $Xt = x$ under a measure $P^\alpha$ that corresponds to an appropriate Escher transform. We therefore obtain the following dampened backward induction:

$$f^N_\alpha(x) = g_\alpha(x),$$

$$f^j_\alpha(x) = g_\alpha(x) \cdot 1_{(-\infty,x^*_j]}(x) + e^{-r\Delta}\phi_\Delta(ia)E_{\Delta,x}[f^{j+1}_\alpha(X_{(j+1)\Delta})] \cdot 1_{(x^*_j,\infty)}(x),$$

$$0 \leq j < N.$$ 

In particular, $x^*_j$ solves the following equation:

$$g_\alpha(x) = e^{-r\Delta}\phi_\Delta(ia)E_{\Delta,x}[f^{j+1}_\alpha(X_{(j+1)\Delta})].$$

In the following, we present the Hilbert transform representation for the above recursion.

### 2.2 Hilbert transform representation

The Hilbert transform of an integrable function $f$ is well defined by the following Cauchy principal value integral (see, e.g., [18]):

$$\mathcal{H}f(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y}dy.$$ 

Denote the Fourier transform of $f$ by $\hat{f}$. Suppose that $f$ is also integrable. A well-known identity in Fourier analysis is crucial for our method is the following: for any $-\infty < l < +\infty$:

$$\mathcal{F}(1_{(l,\infty)} \cdot f)(\xi) = \frac{1}{2} \hat{f}(\xi) + i \frac{1}{2} e^{il\xi} \mathcal{H}(e^{-i\eta l} \hat{f}(\eta))(\xi).$$

We note that the Fourier transform of the conditional expectation in (2.2b) is given by $\hat{f}^j_\alpha(\xi)\phi_\Delta(-\xi)$, where $\phi_\Delta(\cdot) = \phi_\Delta(\cdot + ia)/\phi_\Delta(ia)$ is the characteristic function of $Xt$ under measure $P^\alpha$. Using the above identity, we obtain the following backward induction in the Fourier space for (2.2a)-(2.2c): we start with

$$\hat{f}^N_\alpha(\xi) = \hat{g}_\alpha(\xi).$$

We then find the early exercise boundary $x^*_j$ by solving the following equation using the Newton-Raphson method (note that we use a Fourier inverse representation for the conditional expectation in 2.2c):

$$g_\alpha(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-r\Delta - i\xi x} \hat{f}^{j+1}_\alpha(\xi)\phi_\Delta(-\xi + i\alpha)d\xi.$$ 

Knowing the early exercise boundary, we obtain the following Hilbert transform representation for $\hat{f}^j_\alpha$:

$$\hat{f}^j_\alpha(\xi) = \mathcal{F}(g_\alpha \cdot 1_{(-\infty,x^*_j]})(\xi) \cdot \frac{1}{2} \hat{f}^{j+1}_\alpha(\xi)\phi_\Delta(-\xi + i\alpha) + e^{-r\Delta} \mathcal{H}(e^{-i\eta x^*_j} \hat{f}^{j+1}_\alpha(\eta)\phi_\Delta(-\eta + i\alpha))(\xi).$$

We repeat the above for $1 \leq j < N$. The option value function at time 0 is then computed by:

$$f^0_\alpha(x) = \max(g_\alpha(x),$$

$$0 \leq j < N.$$
\[ \frac{1}{2\pi} e^{-r\Delta} \int_{\mathbb{R}} e^{-i\xi x} f_{\alpha}(\xi) \phi_\Delta(-\xi + ia) d\xi. \]

When needed, (2.3b) can be run one more time for \( j = 0 \) to obtain \( x_0 \). We thus solve (2.3a)-(2.3d) for pricing a Bermudan vanilla put option.

## 3 Discrete approximation

In the above backward induction, we need to evaluate Hilbert transforms and inverse Fourier integrals of certain functions that are analytic in a horizontal strip containing the real axis. Such Hilbert transforms and integrals can be discretized using very simple schemes with remarkable accuracy. It is related to the Whittaker cardinal series (or sinc expansion) for analytic functions. In this section, we briefly review the main results. More details can be found in [19], [8] and [10].

We assume that \( f \) is analytic in a horizontal strip containing the real axis, \( D_d = \{ z \in \mathbb{C} : |\Im(z)| < d \} \), and is well behaved on the boundary of this strip. More specifically, we assume that \( \int_{-d}^{d} f(x + iy) dy \to 0 \) as \( x \to \pm\infty \), and

\[ \| f \|_{L^\infty(D_d)} = \lim_{y \to \pm\infty} \int_{\mathbb{R}} |f(x + iy) + f(x - iy)| dx < \infty. \]

We are interested in computing \( \mathcal{H}f(x) \) and \( I(f) = \int_{-\infty}^{\infty} f(x) dx \). We approximate \( \mathcal{H}f(x) \) by the following discrete Hilbert transform:

\[ \mathcal{H}_h f(x) = \sum_{m=-\infty}^{\infty} f(mh) \frac{1}{\pi(x - mh)/h}, \]

and approximate \( I(f) \) using the trapezoidal sum:

\[ I_h(f) = \sum_{m=-\infty}^{\infty} f(mh) h. \]

Then we have the following error estimates for the discrete Hilbert transform and the trapezoidal sum (see, e.g., Theorem 6.3 in [8]):

\[ \| \mathcal{H}f - \mathcal{H}_h f \|_{L^\infty} \leq \frac{e^{-\pi d/h}}{\pi(1 - e^{-\pi d/h})} \| f \|_{L^\infty(D_d)}, \]

\[ |I(f) - I_h(f)| \leq \frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} \| f \|_{L^\infty(D_d)}. \]

Note that errors of the above discrete approximations converge to zero exponentially fast as the step size \( h \) goes to zero.

We further truncate the infinite sums and approximate \( \mathcal{H}f(x) \) and \( I(f) \) by the following finite sums:

\[ \mathcal{H}_{h,M} f(x) = \sum_{m=-M}^{M} f(mh) \frac{1}{\pi(x - mh)/h}, \]

\[ I_{h,M}(f) = \sum_{m=-M}^{M} f(mh) h. \]

Truncation errors obviously depend on the tail behavior of \( f \). In particular, when

\[ |f(x)| \leq c_1 e^{-c_2|x|^\nu} \]  \hspace{1cm} (3.1)

for some \( c_1, c_2, \nu > 0 \), the above truncation errors also decay exponentially in terms of \( Mh \). One can then select \( h = h(M) \) as a function of \( M \) in certain ways so that the discretization and truncation errors are comparable. The total error then decays exponentially in terms of \( M \), which represents the computational effort. This is convenient in practice since we do not need to select \( h \) and \( M \) separately. More specifically, we have the following rule for \( h = h(M) \) and error estimate for \( \mathcal{H}_{h(M),M} f(x) \) (see Theorem 6.4 in [8]):

\[ h(M) = \left( \frac{\pi d}{c_2} \right)^{1/(1+\nu)} M^{-\nu/(1+\nu)}. \]  \hspace{1cm} (3.2)

\[ \| \mathcal{H}f - \mathcal{H}_{h(M),M} f \|_{L^\infty} \text{ is bounded by} \]

\[ C_1 M^{1/(1+\nu)} \exp(-C_2 M^{\nu/(1+\nu)}). \]  \hspace{1cm} (3.3)

for some \( C_1 > 0 \) and \( C_2 = (\pi d)^{\nu/(1+\nu)} c_2^{1/(1+\nu)} \). Similarly, we have the following rule for \( h = h(M) \) and error estimate for \( I_{h(M),M}(f) \) (see Theorem 6.5 in [8]):

\[ h(M) = \left( \frac{2\pi d}{c_2} \right)^{1/(1+\nu)} M^{-\nu/(1+\nu)}. \]  \hspace{1cm} (3.4)

\[ |I(f) - I_{h(M),M}(f)| \text{ is bounded by} \]

\[ C_3 (1 + M^{(1-\nu)/(1+\nu)}) \exp(-C_4 M^{\nu/(1+\nu)}). \]  \hspace{1cm} (3.5)

for some \( C_3 > 0 \) and \( C_4 = (2\pi d)^{\nu/(1+\nu)} c_2^{1/(1+\nu)} \).

Finally, the Toeplitz matrix multiplication algorithm based on the fast Fourier transform can be used to implement the discrete Hilbert transforms in \( O(M \log(M)) \) operations ([9], [8]). Therefore, the computational cost of our complete algorithm will be \( O(NM \log(M)) \). It is linear in the number of monitoring times, and \( O(M \log(M)) \) in the number of points used to approximate the Hilbert transforms and integrals.
4 Numerical Results

In this section, we present numerical examples on the pricing of Bermudan vanilla puts in the normal inverse Gaussian (NIG) model ([1]). In particular, we verify the exponential convergence of our method. We assume that the risk free interest rate is \( r = 5\% \), the continuous yield the underlying asset is paying is \( q = 2\% \), the option expires in one year \((T = 1)\) and is monitored daily with \( N = 252 \) and \( \Delta = 1/252 \). The strike price is \( K = 100 \). The parameters of the NIG model are \( \alpha_{NIG} = 15 \), \( \beta_{NIG} = -5 \), \( \delta_{NIG} = 0.5 \). All computation is conducted on a Lenovo T400 laptop with 2.53 GHz CPU and 2 GB memory using Matlab version R2009a.

Figure 1: Bermudan puts in the NIG model

![Figure 1: Bermudan puts in the NIG model](image)

We choose the dampening parameter \( \alpha \) so that the convergence rates of the discrete Hilbert transforms and the trapezoidal sums are maximized. The characteristic function \( \phi_t(\cdot) \) in the NIG model satisfies (3.1) with \( \nu = 1 \) and \( c_2 = \delta_{NIG} \Delta \). We then select \( h = h(M) \) according to (3.2). From (3.3), we expect exponential convergence in terms of \( \sqrt{M} \). We price an at-the-money put option: \( S_0 = 100 \). The benchmark option price is computed by taking a large enough \( M \): 6.4895809977. The early exercise boundary at time 0 is \( S^*_0 = 81.1802638151 \). The following figure exhibits the exponential convergence of the pricing error in \( \sqrt{M} \). We observe that our method is very accurate and fast. It takes 3.96 seconds to achieve an accuracy of \( 3 \times 10^{-9} \) with \( M = 7000 \). As for the Newton-Raphson method, as is commonly done, we stop the iteration when the difference between two consecutive approximations is less than a given tolerance level. We use \( 10^{-8} \) in our examples. Actual accuracy turns out to be better than the tolerance level. The Newton-Raphson method takes only 4.08 iterations on average for each time step.

Table 1: American puts in the BSM model

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5 Conclusions

This paper presents a Hilbert transform method for the pricing of Bermudan style contracts in Lévy process models. The method involves a sequential evaluation of Hilbert transforms and inverse Fourier integrals which can be approximated very efficiently with exponentially decaying errors. The computational cost of the method is \( O(NM\log(M)) \), where \( N \) is the number of monitoring times, and \( M \) is the number of points used to approximate the Hilbert transforms or the integrals. American vanilla options can be efficiently and accurately valued using the Richardson extrapolation.

References


