Where are we in CS 440?

• Now leaving: sequential, deterministic reasoning

• Entering: probabilistic reasoning and machine learning
Probability: Review of main concepts (Chapter 13)
Making decisions under uncertainty

• Let action $A_t = \text{leave for airport } t \text{ minutes before flight}$
  – Will $A_t$ succeed, i.e., get me to the airport in time for the flight?
• Problems:
  • Partial observability (road state, other drivers' plans, etc.)
  • Noisy sensors (traffic reports)
  • Uncertainty in action outcomes (flat tire, etc.)
  • Complexity of modeling and predicting traffic
• Hence a non-probabilistic approach either
  • Risks falsehood: “$A_{25}$ will get me there on time,” or
  • Leads to conclusions that are too weak for decision making:
    • $A_{25}$ will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact, etc., etc.
    • $A_{1440}$ will get me there on time but I'll have to stay overnight in the airport
Making decisions under uncertainty

• Suppose the agent believes the following:
  \[ P(A_{25} \text{ gets me there on time}) = 0.04 \]
  \[ P(A_{90} \text{ gets me there on time}) = 0.70 \]
  \[ P(A_{120} \text{ gets me there on time}) = 0.95 \]
  \[ P(A_{1440} \text{ gets me there on time}) = 0.9999 \]

• Which action should the agent choose?
  – Depends on preferences for missing flight vs. time spent waiting
  – Encapsulated by a utility function

• The agent should choose the action that maximizes the expected utility:
  \[ P(A_t \text{ succeeds}) \times U(A_t \text{ succeeds}) + P(A_t \text{ fails}) \times U(A_t \text{ fails}) \]
Making decisions under uncertainty

- More generally: the expected utility of an action is defined as:

\[ EU(a) = \sum_{\text{outcomes of } a} P(\text{outcome} \mid a) \cdot U(\text{outcome}) \]

- **Utility theory** is used to represent and infer preferences
- **Decision theory** = probability theory + utility theory
Monty Hall problem

You’re a contestant on a game show. You see three closed doors, and behind one of them is a prize. You choose one door, and the host opens one of the other doors and reveals that there is no prize behind it. Then he offers you a chance to switch to the remaining door. Should you take it?

http://en.wikipedia.org/wiki/Monty_Hall_problem
Monty Hall problem

• With probability 1/3, you picked the correct door, and with probability 2/3, picked the wrong door. If you picked the correct door and then you switch, you lose. If you picked the wrong door and then you switch, you win the prize.

• Expected utility of switching:
  \[ EU(\text{Switch}) = (1/3) \times 0 + (2/3) \times \text{Prize} \]

• Expected utility of not switching:
  \[ EU(\text{Not switch}) = (1/3) \times \text{Prize} + (2/3) \times 0 \]
Where do probabilities come from?

• **Frequentism**
  – Probabilities are relative frequencies
  – For example, if we toss a coin many times, \( P(\text{heads}) \) is the proportion of the time the coin will come up heads
  – But what if we’re dealing with events that only happen once?
    • E.g., what is the probability that Team X will win the Superbowl this year?
    • “Reference class” problem

• **Subjectivism**
  – Probabilities are degrees of belief
  – But then, how do we assign belief values to statements?
  – What would constrain agents to hold consistent beliefs?
Probabilities and rationality

• Why should a rational agent hold beliefs that are consistent with axioms of probability?
  – For example, \( P(A) + P(\neg A) = 1 \)

• If an agent has some degree of belief in proposition \( A \), he/she should be able to decide whether or not to accept a bet for/against \( A \) (De Finetti, 1931):
  – If the agent believes that \( P(A) = 0.4 \), should he/she agree to bet $4 that \( A \) will occur against $6 that \( A \) will not occur?

• **Theorem**: An agent who holds beliefs inconsistent with axioms of probability can be convinced to accept a combination of bets that is guaranteed to lose them money
Random variables

• We describe the (uncertain) state of the world using *random variables*
  - Denoted by capital letters
    - **R**: Is it raining?
    - **W**: What’s the weather?
    - **D**: What is the outcome of rolling two dice?
    - **S**: What is the speed of my car (in MPH)?

• Just like variables in CSPs, random variables take on values in a *domain*
  - Domain values must be *mutually exclusive* and *exhaustive*
    - **R** in \{True, False\}
    - **W** in \{Sunny, Cloudy, Rainy, Snow\}
    - **D** in \{(1,1), (1,2), \ldots (6,6)\}
    - **S** in [0, 200]
Events

- Probabilistic statements are defined over events, or sets of world states
  - “It is raining”
  - “The weather is either cloudy or snowy”
  - “The sum of the two dice rolls is 11”
  - “My car is going between 30 and 50 miles per hour”

- Events are described using propositions about random variables:
  - $R = \text{True}$
  - $W = \text{"Cloudy"} \lor W = \text{"Snowy"}$
  - $D \in \{(5,6), (6,5)\}$
  - $30 \leq S \leq 50$

- Notation: $P(A)$ is the probability of the set of world states in which proposition $A$ holds
Kolmogorov’s axioms of probability

• For any propositions (events) A, B
  ▪ 0 ≤ P(A) ≤ 1
  ▪ P(True) = 1 and P(False) = 0
  ▪ P(A ∨ B) = P(A) + P(B) − P(A ∧ B)
    – Subtraction accounts for double-counting

• Based on these axioms, what is P(¬A)?

• These axioms are sufficient to completely specify probability theory for discrete random variables
  • For continuous variables, need density functions
Atomic events

- **Atomic event**: a complete specification of the state of the world, or a complete assignment of domain values to all random variables
  - Atomic events are mutually exclusive and exhaustive

- E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are four distinct atomic events:

  \[
  \begin{align*}
  Cavity &= false \land Toothache = false \\
  Cavity &= false \land Toothache = true \\
  Cavity &= true \land Toothache = false \\
  Cavity &= true \land Toothache = true
  \end{align*}
  \]
Joint probability distributions

- A **joint distribution** is an assignment of probabilities to every possible atomic event.

<table>
<thead>
<tr>
<th>Atomic event</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Cavity} = \text{false} \land \text{Toothache} = \text{false}$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{false} \land \text{Toothache} = \text{true}$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{true} \land \text{Toothache} = \text{false}$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{true} \land \text{Toothache} = \text{true}$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

- Why does it follow from the axioms of probability that the probabilities of all possible atomic events must sum to 1?
Joint probability distributions

- A **joint distribution** is an assignment of probabilities to every possible atomic event.
- Suppose we have a joint distribution of $n$ random variables with domain sizes $d$.
  - What is the size of the probability table?
  - Impossible to write out completely for all but the smallest distributions.
Notation

- $P(X_1 = x_1, X_2 = x_2, ..., X_n = x_n)$ refers to a single entry (atomic event) in the joint probability distribution table
  - Shorthand: $P(x_1, x_2, ..., x_n)$
- $P(X_1, X_2, ..., X_n)$ refers to the entire joint probability distribution table
- $P(A)$ can also refer to the probability of an event
  - E.g., $X_1 = x_1$ is an event
Marginal probability distributions

- From the joint distribution $P(X,Y)$ we can find the **marginal distributions** $P(X)$ and $P(Y)$

<table>
<thead>
<tr>
<th>P(Cavity, Toothache)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Cavity} = \text{false} \land \text{Toothache} = \text{false}$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{false} \land \text{Toothache} = \text{true}$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{true} \land \text{Toothache} = \text{false}$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{true} \land \text{Toothache} = \text{true}$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>P(Cavity)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Cavity} = \text{false}$</td>
<td>?</td>
</tr>
<tr>
<td>$\text{Cavity} = \text{true}$</td>
<td>?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>P(Toothache)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Toothache} = \text{false}$</td>
<td>?</td>
</tr>
<tr>
<td>$\text{Toothache} = \text{true}$</td>
<td>?</td>
</tr>
</tbody>
</table>
Marginal probability distributions

- From the joint distribution $P(X,Y)$ we can find the **marginal distributions** $P(X)$ and $P(Y)$
- To find $P(X = x)$, sum the probabilities of all atomic events where $X = x$:

$$P(X = x) = P((X = x \land Y = y_1) \lor \ldots \lor (X = x \land Y = y_n))$$

$$= P((x, y_1) \lor \ldots \lor (x, y_n)) = \sum_{i=1}^{n} P(x, y_i)$$

- This is called **marginalization** (we are marginalizing out all the variables except $X$)
Conditional probability

- Probability of cavity given toothache:
  \( P(Cavity = true \mid Toothache = true) \)

- For any two events A and B, \( P(A \mid B) = \)
Conditional probability

\[
P(Cavity = true \mid Toothache = false) = \frac{0.05}{0.85} = 0.059
\]

\[
P(Cavity = false \mid Toothache = true) = \frac{0.1}{0.15} = 0.667
\]
Conditional distributions

- A conditional distribution is a distribution over the values of one variable given fixed values of other variables.

<table>
<thead>
<tr>
<th>P(Cavity, Toothache)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavity = false ∧ Toothache = false</td>
<td>0.8</td>
</tr>
<tr>
<td>Cavity = false ∧ Toothache = true</td>
<td>0.1</td>
</tr>
<tr>
<td>Cavity = true ∧ Toothache = false</td>
<td>0.05</td>
</tr>
<tr>
<td>Cavity = true ∧ Toothache = true</td>
<td>0.05</td>
</tr>
</tbody>
</table>

| P(Cavity | Toothache = true) |         |
|----------|--------------------|
| Cavity = false | 0.667       |
| Cavity = true   | 0.333       |

| P(Cavity | Toothache = false) |         |
|----------|---------------------|
| Cavity = false | 0.941       |
| Cavity = true   | 0.059       |

| P(Toothache | Cavity = true) |         |
|-------------|----------------|
| Toothache= false | 0.5     |
| Toothache = true | 0.5     |

| P(Toothache | Cavity = false) |         |
|-------------|-----------------|
| Toothache= false | 0.889     |
| Toothache = true | 0.111     |
Normalization trick

- To get the whole conditional distribution $P(X \mid Y = y)$ at once, select all entries in the joint distribution table matching $Y = y$ and renormalize them to sum to one.

<table>
<thead>
<tr>
<th>P(Cavity, Toothache)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$Cavity = false \land Toothache = false$</td>
<td>0.8</td>
</tr>
<tr>
<td>$Cavity = false \land Toothache = true$</td>
<td>0.1</td>
</tr>
<tr>
<td>$Cavity = true \land Toothache = false$</td>
<td>0.05</td>
</tr>
<tr>
<td>$Cavity = true \land Toothache = true$</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Select

<table>
<thead>
<tr>
<th>Toothache, Cavity = false</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Toothache = false</td>
<td>0.8</td>
</tr>
<tr>
<td>Toothache = true</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Renormalize

<table>
<thead>
<tr>
<th>P(Toothache $\mid$ Cavity = false)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Toothache = false</td>
<td>0.889</td>
</tr>
<tr>
<td>Toothache = true</td>
<td>0.111</td>
</tr>
</tbody>
</table>
Normalization trick

- To get the whole conditional distribution $P(X \mid Y = y)$ at once, select all entries in the joint distribution table matching $Y = y$ and renormalize them to sum to one.
- Why does it work?

$$\frac{P(x, y)}{\sum_{x'} P(x', y)} = \frac{P(x, y)}{P(y)}$$

by marginalization.
Product rule

• Definition of conditional probability: \( P(A \mid B) = \frac{P(A, B)}{P(B)} \)

• Sometimes we have the conditional probability and want to obtain the joint:

\[
P(A, B) = P(A \mid B)P(B) = P(B \mid A)P(A)
\]
Chain rule

- Product rule:

\[ P(A, B) = P(A \mid B)P(B) = P(B \mid A)P(A) \]

- Chain rule:

\[
P(A_1, \ldots, A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1, A_2) \ldots P(A_n \mid A_1, \ldots, A_{n-1})
\]

\[
= \prod_{i=1}^{n} P(A_i \mid A_1, \ldots, A_{i-1})
\]
Independence

- Two events $A$ and $B$ are *independent* if and only if
  \[ P(A \land B) = P(A, B) = P(A) \cdot P(B) \]
  - In other words, $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$
  - This is an important simplifying assumption for modeling, e.g., *Toothache* and *Weather* can be assumed to be independent

- Are two *mutually exclusive* events independent?
  - No, but for mutually exclusive events we have
    \[ P(A \lor B) = P(A) + P(B) \]
Independence

- Two events A and B are \textit{independent} if and only if
  \[ P(A \land B) = P(A, B) = P(A) P(B) \]
  - In other words, \( P(A \mid B) = P(A) \) and \( P(B \mid A) = P(B) \)
  - This is an important simplifying assumption for modeling, e.g., \textit{Toothache} and \textit{Weather} can be assumed to be independent

- \textbf{Conditional independence}: A and B are \textit{conditionally independent} given C iff
  \[ P(A \land B \mid C) = P(A \mid C) P(B \mid C) \]
  - Equivalently:
    \[ P(A \mid B, C) = P(A \mid C) \text{ or } P(B \mid A, C) = P(B \mid C) \]
Conditional independence: Example

- **Toothache**: boolean variable indicating whether the patient has a toothache
- **Cavity**: boolean variable indicating whether the patient has a cavity
- **Catch**: whether the dentist's probe catches in the cavity

If the patient has a cavity, the probability that the probe catches in it doesn't depend on whether he/she has a toothache

\[
P(\text{Catch} | \text{Toothache}, \text{Cavity}) = P(\text{Catch} | \text{Cavity})
\]

Therefore, **Catch** is conditionally independent of **Toothache** given **Cavity**

Likewise, **Toothache** is conditionally independent of **Catch** given **Cavity**

\[
P(\text{Toothache} | \text{Catch}, \text{Cavity}) = P(\text{Toothache} | \text{Cavity})
\]

Equivalent statement:

\[
P(\text{Toothache}, \text{Catch} | \text{Cavity}) = P(\text{Toothache} | \text{Cavity}) P(\text{Catch} | \text{Cavity})
\]
Conditional independence: Example

• How many numbers do we need to represent the joint probability table $P(Toothache, Cavity, Catch)$?
  $2^3 - 1 = 7$ independent entries
• Write out the joint distribution using chain rule:

\[
P(Toothache, Catch, Cavity) = P(Cavity) \, P(Catch | Cavity) \, P(Toothache | Catch, Cavity)
= P(Cavity) \, P(Catch | Cavity) \, P(Toothache | Cavity)
\]
• How many numbers do we need to represent these distributions?
  $1 + 2 + 2 = 5$ independent numbers
• In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in $n$ to linear in $n$
The Birthday problem

- We have a set of $n$ people. What is the probability that two of them share the same birthday?
- Easier to calculate the probability that $n$ people do not share the same birthday

\[
P(B_1, \ldots B_n \text{ distinct}) \\
= P(B_n \text{ distinct from } B_1, \ldots B_{n-1} \mid B_1, \ldots B_{n-1} \text{ distinct}) \\
P(B_1, \ldots B_{n-1} \text{ distinct}) \\
= \prod_{i=1}^{n} P(B_i \text{ distinct from } B_1, \ldots B_{i-1} \mid B_1, \ldots B_{i-1} \text{ distinct})
\]
The Birthday problem

\[ P(B_1, \ldots B_n \text{ distinct}) \]

\[ = \prod_{i=1}^{n} P(B_i \text{ distinct from } B_1, \ldots B_{i-1} \mid B_1, \ldots B_{i-1} \text{ distinct}) \]

\[ P(B_i \text{ distinct from } B_1, \ldots , B_{i-1} \mid B_1, \ldots , B_{i-1} \text{ distinct}) = \frac{365 - i + 1}{365} \]

\[ P(B_1, \ldots , B_n \text{ distinct}) = \frac{365 \times 364 \times \ldots \times 365 - n + 1}{365^{n}} \]

\[ P(B_1, \ldots , B_n \text{ not distinct}) = 1 - \frac{365 \times 364 \times \ldots \times 365 - n + 1}{365^{n}} \]
The Birthday problem

• For 23 people, the probability of sharing a birthday is above 0.5!