A Proof of Dudley’s Convex Approximation

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Abstract

We provide a self contained proof of a result of Dudley [Dud74], which shows that a bounded convex-body in $\mathbb{R}^d$ can be $\varepsilon$-approximated, by the intersection of $O_d(\varepsilon^{-(d-1)/2})$ halfspaces, where $O_d$ hides constants that depends on $d$.

1. Statement and proof

For a convex body $C \subseteq \mathbb{R}^d$, let $C_{\varepsilon}$ denote the set of all points in $\mathbb{R}^d$ in distance at most $\leq \varepsilon$ from $C$. In particular, $C \subseteq C_{\varepsilon}$, and the Hausdorff distance between $C$ and $C_{\varepsilon}$ is $\varepsilon$.

**Theorem 1.1 ([Dud74]).** Let $C$ be a (closed) convex body in $\mathbb{R}^d$, containing the unit ball of radius one centered at the origin, such that $C$ is contained in a ball of radius $d$ centered at the origin. For a parameter $\varepsilon > 0$, one can compute a convex body $D$, which is the intersection of $O_d(1/\varepsilon^{(d-1)/2})$ halfspaces, such that $C \subseteq D \subseteq C_{\varepsilon}$.

**Proof:** Let $S$ be the sphere of radius $2d$ centered at the origin, and let $Q$ be a maximal $\delta$-packing of $S$, where $\delta = \sqrt{d\varepsilon}/8$. We remind the reader that a set $Q \subseteq S$ is a $\delta$-packing, if

(i) for any point $p \in S$, there is a point $q \in Q$, such that $\|p - q\| \leq \delta$, and
(ii) for any two points $q, q' \in Q$, we have that $\|q - q'\| \leq \delta$.

In particular, it is easy to verify that $|Q| = O_d((d/\delta)^{d-1}) = O_d(\varepsilon^{-(d-1)/2})$.

Next, for every point $q \in Q$, let $n(q)$ be its nearest neighbor in $C$ (which naturally lies on $\partial C$), and consider the halfspace that passes through $n(q)$, contains $C$, and is orthogonal to the vector $q - n(q)$. Let $h_C(q)$ denote this halfspace. Let $D = \bigcap_{q \in Q} h_C(q)$. We claim that $D$ is the desired approximation.

First, it is clear that $C \subseteq D$. As for the other direction, consider any point $p \in \partial C$, and consider a normal to $C$ at $p$, denoted by $v$. Consider the ray emanating from $p$ in the direction of $v$. It hits $S$ at a point $p'$, and let $q \in Q$, be the nearest point in the packing $Q$ to it. Next, consider $n(q)$. It is easy to verify that $\|p - n(q)\| \leq \|p' - q\| \leq \delta$ (because projecting to nearest-neighbor is a contraction).

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We are interested in angle between \( v \) and \( n(q) - q \). To this end, observe that \( \|p' - q\| \leq \delta, \|p - n(q)\| \leq \delta, \|p - p'\| \geq d \), and \( \|q - n(q)\| \geq d \). Let \( x = q - n(q) \) and \( y = p' - p \). Notice that

\[
\ell = \|x - y\| = \|q - n(q) - (p' - p)\| = \|(p - n(q)) + (q - p')\| \\
\leq \|p - n(q)\| + \|q - p'\| \leq 2\delta.
\]

Observe that \( \|x\| \leq \text{radius}(S) = 2d \) and \( \|y\| \leq 2d \). Let \( \triangle \) be the triangle formed by the origin, \( x \) and \( y \). The height of \( \triangle \) at \( x \) is bounded by \( \ell \). As such, \( \text{area}(\triangle) \leq \frac{1}{2} \max(\|x\|, \|y\|) \text{height}(\triangle) \leq d \text{height}(\triangle) \leq d\ell \leq 2d\delta \).

Let \( \gamma \) denote the angle between \( x \) and \( y \). We have that

\[
\text{area}(\triangle) = \frac{1}{2} \|x\| \|y\| \sin \gamma \leq 2d\delta \quad \Rightarrow \quad \sin \gamma \leq \frac{4\delta}{d}.
\]

Let \( h \) be the three dimensional affine subspace that is spanned by the vectors \( x, y, p - n(q) \), and passes through \( n(q) \). Clearly, \( p \in h \). Now, \( H_q = h \cap h_C(q) \) and \( H_{p'} = h \cap h_C(p') \) are two halfspaces contained in \( h \). The angle between their bounding planes is exactly \( \gamma \) (as their normals are \( x \) and \( y \)). In particular, let \( f \subseteq h \) be the two dimensional plane that contains the points \( n(q), q, p \). Let \( \ell \) be the line \( \partial H_q \cap \partial H_{p'} \), and let \( t \) be the intersection of \( f \) with \( \ell \).

The distance of \( p \) from \( \partial h_C(q) \) bounds the distance of \( p \) from the boundary of \( D \). This distance in turn is bounded by the distance from \( p \) to the line \( \ell' \) spanned by \( n(q) \) and \( t \). Let \( \beta \) be the angle between \( \ell' \) and \( pt \) (see figure). It is easy to verify that as \( f \) contains the vector \( x \), this implies that \( \beta \leq \gamma \). This in turn implies that \( \angle tn(q)p \leq \beta \leq \gamma \). Using the packing property that \( \delta \leq \sqrt{d\varepsilon/8} \), we have

\[
\text{dist}(p, \partial D) \leq \text{dist}(p, \ell') \leq \|p - n(q)\| \sin \beta \leq \delta \sin \gamma \leq \delta \frac{4\delta}{d} \leq \frac{\varepsilon}{2}.
\]

The distance of any point of \( \partial C_{\bar{\beta}\varepsilon} \) from \( C \) is at least \( \varepsilon/2 \). It follows that \( D \subseteq C_{\bar{\beta}\varepsilon} \).  

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References