9 Minimum Cuts in Surface Graphs

The input to the classical minimum cut problem is a graph \( G = (V, E) \), a non-negative capacity function \( c : E \rightarrow \mathbb{R} \), and two vertices \( s \) and \( t \). The goal is to find a subset \( X \) of the edges of minimum total capacity (minimum), such that \( s \) and \( t \) are in different components of \( G \setminus X \) (cut). Minimum cuts and maximum flows have been of intense research since their introduction by Ford, Fulkerson, and other researchers at the RAND Corporation in the 1950s [6, 2, 3, 4].

For graphs embedded on surfaces, this classical problem has the following dual formulation: Given a graph \( G^* \) embedded on \( \Sigma \), a non-negative cost function \( c^* : E^* \rightarrow \mathbb{R} \), and two faces \( s^* \) and \( t^* \), find a subgraph \( X^* \) of \( G^* \) of minimum total cost, such that \( s^* \) and \( t^* \) are in different components of \( \Sigma \setminus X^* \). We have already seen one special case of this problem. When \( \Sigma \) is the sphere, the subgraph \( X^* \) is precisely the shortest non-contractible cycle in the annulus \( \Sigma \setminus (s^* \cup t^*) \). The algorithm of Reif [8] and Frederickson [5] discussed in the previous lecture solves this problem in \( O(n \log n) \) time. In fact, the true goal of Reif and Frederickson’s work was to solve the dual problem: finding minimum cuts in undirected planar graphs.

Erin Chambers, Amir Nayyeri, and I recently discovered an algorithm to find minimum cuts in arbitrary surface graphs in \( O(g \log n) \) time [1]; I’ll describe our algorithm in this lecture. As one might expect from the running time, our algorithm is a generalization of Kutz’s technique for finding shortest non-contractible cycles [7]. For surfaces other than the sphere, the dual minimum cut is not necessarily a single cycle, but the union of up to \( g + 1 \) edge-disjoint cycles. Our algorithm enumerates all possible homotopy types of all cycles that could be part of the dual minimum cut, computes the shortest cycle in each homotopy class using Kutz’s algorithm, and then assembles the dual minimum cut from those shortest cycles.

I did not prepare separate notes for this lecture; for further details, please see our paper [1].

References


