1. Clearly indicate the edges of the following spanning trees of the weighted graph pictured below. (Pretend that the person grading your exam has bad eyesight.) Some of these subproblems have more than one correct answer. Yes, that edge on the right has negative weight.

(a) A depth-first spanning tree rooted at \( s \)
(b) A breadth-first spanning tree rooted at \( s \)
(c) A shortest-path tree rooted at \( s \)
(d) A minimum spanning tree

2. Farmers Boggis, Bunce, and Bean have set up an obstacle course for Mr. Fox. The course consists of a row of \( n \) booths, each with an integer painted on the front with bright red paint, which could be positive, negative, or zero. Let \( A[i] \) denote the number painted on the front of the \( i \)th booth. Everyone has agreed to the following rules:

- At each booth, Mr. Fox must say either “Ring!” or “Ding!”
- If Mr. Fox says “Ring!” at the \( i \)th booth, he earns a reward of \( A[i] \) chickens. (If \( A[i] < 0 \), Mr. Fox pays a penalty of \( -A[i] \) chickens.)
- If Mr. Fox says “Ding!” at the \( i \)th booth, he pays a penalty of \( A[i] \) chickens. (If \( A[i] < 0 \), Mr. Fox earns a reward of \( -A[i] \) chickens.)
- Mr. Fox is forbidden to say the same word more than three times in a row. For example, if he says “Ring!” at booths 6, 7, and 8, then he must say “Ding!” at booth 9.
- All accounts will be settled at the end; Mr. Fox does not actually have to carry chickens through the obstacle course.
- If Mr. Fox violates any of the rules, or if he ends the obstacle course owing the farmers chickens, the farmers will shoot him.

Describe and analyze an algorithm to compute the largest number of chickens that Mr. Fox can earn by running the obstacle course, given the array \( A[1..n] \) of booth numbers as input.

3. Let \( G \) be a directed graph with weighted edges, and let \( s \) be a vertex of \( G \). Suppose every vertex \( v \neq s \) stores a pointer \( \text{pred}(v) \) to another vertex in \( G \). Describe and analyze an algorithm to determine whether these predecessor pointers correctly define a single-source shortest path tree rooted at \( s \). Do not assume that \( G \) has no negative cycles.
4. An array $A[0..n-1]$ of $n$ distinct numbers is **bitonic** if there are unique indices $i$ and $j$ such that $A[(i-1) \mod n] < A[i] > A[(i+1) \mod n]$ and $A[(j-1) \mod n] > A[j] < A[(j+1) \mod n]$. In other words, a bitonic sequence either consists of an increasing sequence followed by a decreasing sequence, or can be circularly shifted to become so. For example,

$$
\begin{array}{cccccccc}
4 & 6 & 9 & 8 & 7 & 5 & 1 & 2 & 3 \\
3 & 6 & 9 & 8 & 7 & 5 & 1 & 2 & 4 \\
\end{array}
$$

is bitonic, but

$$
\begin{array}{cccccccc}
3 & 6 & 9 & 8 & 7 & 5 & 1 & 2 & 4 \\
\end{array}
$$

is not bitonic.

Describe and analyze an algorithm to find the index of the *smallest* element in a given bitonic array $A[0..n-1]$ in $O(\log n)$ time. You may assume that the numbers in the input array are distinct. For example, given the first array above, your algorithm should return 6, because $A[6] = 1$ is the smallest element in that array.

5. Suppose we are given an undirected graph $G$ in which every vertex has a positive weight.

(a) Describe and analyze an algorithm to find a *spanning tree* of $G$ with minimum total weight. (The total weight of a spanning tree is the sum of the weights of its vertices.)

(b) Describe and analyze an algorithm to find a *path* in $G$ from one given vertex $s$ to another given vertex $t$ with minimum total weight. (The total weight of a path is the sum of the weights of its vertices.)