1. Suppose we are given a directed acyclic graph $G$ with labeled vertices. Every path in $G$ has a label, which is a string obtained by concatenating the labels of its vertices in order. Recall that a palindrome is a string that is equal to its reversal.

Describe and analyze an algorithm to find the length of the longest palindrome that is the label of a path in $G$. For example, given the graph below, your algorithm should return the integer 6, which is the length of the palindrome $\text{HANNAH}$.

![Graph Diagram]

2. Let $G$ be a connected directed graph that contains both directions of every edge; that is, if $u \rightarrow v$ is an edge in $G$, its reversal $v \rightarrow u$ is also an edge in $G$. Consider the following non-standard traversal algorithm.

\[\text{SPAGHETTI}(v):\]
\[\text{mark } v \quad \text{\texttt{\{"visit" }v\text{"\}}}\]
\[\text{if there is a white arc } v \rightarrow w\]
\[\quad \text{if } w \text{ is unmarked}\]
\[\quad \text{color } w \rightarrow v \text{ green}\]
\[\quad \text{color } v \rightarrow w \text{ red} \quad \text{\texttt{\{"traverse" }v \rightarrow w\text{"\}}}\]
\[\text{SPAGHETTI}(w)\]
\[\text{else if there is a green arc } v \rightarrow w\]
\[\quad \text{color } v \rightarrow w \text{ red} \quad \text{\texttt{\{"traverse" }v \rightarrow w\text{"\}}}\]
\[\text{SPAGHETTI}(w)\]
\[\text{\texttt{(\{\text{\textit{else every arc } v \rightarrow w \text{ is red, so halt}}\})}}\]

We informally say that this algorithm “visits” vertex $v$ every time it marks $v$, and it “traverses” edge $v \rightarrow w$ when it colors that edge red. Unlike our standard graph-traversal algorithms, SPAGHETTI may (in fact, will) mark/visit each vertex more than once.

The following series of exercises leads to a proof that SPAGHETTI traverses each directed edge of $G$ exactly once. Most of the solutions are very short.

(a) Prove that no directed edge in $G$ is traversed more than once.
(b) When the algorithm visits a vertex $v$ for the $k$th time, exactly how many edges into $v$ are red, and exactly how many edges out of $v$ are red? [Hint: Consider the starting vertex $s$ separately from the other vertices.]
(c) Prove each vertex \( v \) is visited at most \( \deg(v) \) times, except the starting vertex \( s \), which is visited at most \( \deg(s) + 1 \) times. This claim immediately implies that \( \text{SPAGHETTI}\text{TRAVERSAL}(G) \) terminates.

(d) Prove that when \( \text{SPAGHETTI}\text{TRAVERSAL}(G) \) ends, the last visited vertex is the starting vertex \( s \).

(e) For every vertex \( v \) that \( \text{SPAGHETTI}\text{TRAVERSAL}(G) \) visits, prove that all edges incident to \( v \) (either in or out) are red when \( \text{SPAGHETTI}\text{TRAVERSAL}(G) \) halts. \textit{[Hint: Consider the vertices in the order that they are marked for the first time, starting with} \( s \), \textit{and prove the claim by induction.]}

(f) Prove that \( \text{SPAGHETTI}\text{TRAVERSAL}(G) \) visits every vertex of \( G \).

(g) Finally, prove that \( \text{SPAGHETTI}\text{TRAVERSAL}(G) \) traverses every edge of \( G \) exactly once.