Multi-commodity Flows and Cuts in *Polymatroidal* Networks

Chandra Chekuri
*Univ. of Illinois, Urbana-Champaign*

Joint work with
Sreeram Kannan, Adnan Raja, Pramod Viswanath
(UIUC ECE Department)

Max-flow Min-cut Theorem

[Ford-Fulkerson, Menger]

G = (V, E) directed graph with non-negative edge-capacities

max s\-t flow value equal to min s\-t cut value

if capacities integral max flow can be chosen to be integral
Multi-commodity Flows

Several pairs \((s_1,t_1),\ldots,(s_k,t_k)\) jointly use the network capacity to route their flow

\[ f_i(e) : \text{flow for pair } i \text{ on edge } e \]

\[ \sum_i f_i(e) \leq c(e) \quad \text{for all } e \]
$f_i(e)$: flow for pair $i$ on edge $e$

$\sum_i f_i(e) \leq c(e)$ for all $e$

$max \sum_i val(f_i)$ (max throughput flow)
Max Throughput Flow and Min Multicut

\[ f_i(e) : \text{flow for pair } i \text{ on edge } e \]

\[ \sum_i f_i(e) \leq c(e) \quad \text{for all } e \]

\[ \max \sum_i \text{val}(f_i) \quad \text{(max throughput flow)} \]

\[ \textbf{Multicut: set of edges whose removal disconnects all pairs} \]

Max Throughput Flow \( \leq \) Min Multicut Capacity
Max Concurrent Flow and Min Sparsest Cut

\[ f_i(e) : \text{flow for pair } i \text{ on edge } e \]

\[ \sum_i f_i(e) \leq c(e) \text{ for all } e \]

\[ \text{val}(f_i) \geq \lambda D_i \text{ for all } i \]

\[ \max \lambda \text{ (max concurrent flow)} \]
Max Concurrent Flow and Min Sparsest Cut

\[ f_i(e) : \text{flow for pair } i \text{ on edge } e \]

\[ \sum_i f_i(e) \leq c(e) \quad \text{for all } e \]

\[ \text{val}(f_i) \geq \lambda D_i \quad \text{for all } i \]

\[ \max \lambda \quad (\text{max concurrent flow}) \]

**Sparsity of cut** = capacity of cut / demand separated by cut

Max Concurrent Flow \( \leq \) Min Sparsity
Flow-Cut Gap: Undir graphs

[Leighton-Rao’88] examples via expanders to show
Max Throughput Flow $\leq O(1/\log k)$ Min Multicut
Max Concurrent Flow $\leq O(1/\log k)$ Min Sparsity

$k = \Theta(n^2)$ in expander examples
Flow-Cut Gap: Undir graphs

[Leighton-Rao’88] for product multi-commodity flow
Max Concurrent Flow $\geq \Omega \left(1/\log k\right)$ Min Sparsity

[Garg-Vazirani-Yannakakis’93]
Max Throughput Flow $\geq \Omega \left(1/\log k\right)$ Min Multicut

[Linial-London-Rabinovich’95,Aumann-Rabani’95]
Max Concurrent Flow $\geq \Omega \left(1/\log k\right)$ Min Sparsity
Flow-Cut Gap: Undir graphs
Node Capacities

[Garg-Vazirani-Yannakakis’93]
Max Throughput Flow $\geq \Omega(1/\log k)$ Min Multicut

[Feige-Hajiaghayi-Lee’05]
Max Concurrent Flow $\geq \Omega(1/\log k)$ Min Sparsity
Flow-Cut Gap: Dir graphs

[Saks-Samorodnitsky-Zosin’04]
Max Throughput Flow $\leq O(1/k)$ Min Multicut

[Chuzhoy-Khanna’07]
Max Throughput Flow $\leq O(1/n^{1/7})$ Min Multicut

[Agrawal-Alon-Charikar’07]
Max Throughput Flow $\geq \Omega(1/n^{11/23})$ Min Multicut
$\geq 1/k$ Min Multicut (trivial)
Flow-Cut Gap: Dir graphs

**Symmetric demands:** \((s_i, t_i)\) and \((t_i, s_i)\) for each pair and cut has to separate only one of the two

[Klein-Plotkin-Rao-Tardos’97]

Max Throughput Flow \(\geq \Omega(1/\log^2 k)\) Min Multicut

Max Concurrent Flow \(\geq \Omega(1/\log^3 k)\) Min Sparsity

[Even-Naor-Rao-Schieber’95]

Max Throu. Flow \(\geq \Omega(1/\log n \log \log \log n)\) Min Multicut
Flow-Cut Gaps: Summary

$k$ pairs in a graph $G=(V,E)$

- $\Theta(\log k)$ for undir graphs (edge and node capacities)
  - Throughput Flow vs Multicut
  - Concurrent Flow vs Sparsest Cut

- $O(poly\log(k))$ for dir graph with symmetric demands

- Polynomial-factor lower bounds for dir graphs
Polymatroidal Networks

Capacity of edges incident to $v$ jointly constrained by a polymatroid (monotone non-neg submodular set func)

$$\sum_{i \in S} c(e_i) \leq f(S) \text{ for every } S \subseteq \{1,2,3,4\}$$
**Question:** What is the *information theoretic capacity* of a network?

Given $G=(V,E)$ and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ and rates/demands $D_1,\ldots,D_k$: can the pairs use the network to successfully transmit information at these rates?

- Can use routing, (network) coding, and any other scheme ...
- Network coding [Ahlswede-Cai-Li-Yeung’00]
Network Information Theory: Cut-Set Bound

Max Concurrent Rate $\leq$ Min Sparsity
Network Information Theory

Max Concurrent Rate $\leq$ Min Sparsity

- In undirected graphs routing is near-optimal (within log factors). Follows from flow-cut gap upper bounds.
- In directed graphs routing can be very far from optimal.
- In directed graphs routing far from optimal even for multicast.
- Capacity of networks poorly understood.
Capacity of Wireless Networks

[Diagram showing components of a wireless network, including DSL/Cable Modem, Wireless Router, Wireless Repeater, and Computers with network adapters.]
Capacity of wireless networks

Major issues to deal with:

• interference due to broadcast nature of medium
• noise
Capacity of wireless networks

Recent work: understand/model/approximate wireless networks via wireline networks

- Linear deterministic networks [Avestimehr-Diggavi-Tse’09]
  - Unicast/multicast (single source). Connection to polylinking systems and submodular flows [Goemans-Iwata-Zenklusen’09]

- Polymatroidal networks [Kannan-Viswanath’11]
  - Multiple unicast.
Directed Polymatroidal Networks

[Lawler-Martel’82, Hassin’79]

Directed graph $G=(V,E)$

For each node $v$ two polymatroids

- $\rho_v^-$ with ground set $\delta^-(v)$
- $\rho_v^+$ with ground set $\delta^+(v)$

\[
\sum_{e \in S} f(e) \leq \rho_v^-(S) \quad \text{for all } S \subseteq \delta^-(v)
\]
\[
\sum_{e \in S} f(e) \leq \rho_v^+(S) \quad \text{for all } S \subseteq \delta^+(v)
\]
Flow from $s$ to $t$: “standard flow” with polymatroidal capacity constraints
What is the cap. of a cut?

Assign each edge \((a,b)\) of cut to either \(a\) or \(b\)

Value = sum of function values on assigned sets

Optimize over all assignments

\[
\min \{1+1+1, 1.2+1, 1.6+1\}
\]
What is the cap. of a cut?

Other possibilities and why they don’t work

• assign edges to both ends and take average
• assign edges to both ends and take minimum
Maxflow-Mincut Theorem

[Lawler-Martel’82, Hassin’79]

**Theorem:** In a directed polymatroidal network the max s-t flow is equal to the min s-t cut value.

Model equivalent to submodular-flow model of [Edmonds-Giles’77] that can derive as special cases

- polymatroid intersection theorem
- maxflow-mincut in standard network flows
- Lucchesi-Younger theorem
Undirected Polymatroidal Networks

“New” model:

Undirected graph $G=(V,E)$

For each node $v$ single polymatroids

- $\rho_v$ with ground set $\delta(v)$

$$\sum_{e \in S} f(e) \leq \rho_v(S) \text{ for all } S \subseteq \delta(v)$$

Note: maxflow-mincut does not hold, only within factor of 2!
Why Undirected Polymatroidal Networks?

- captures node-capacitated flows in undirected graphs
- within factor of 2 approximates bi-directed polymatroidal networks relevant to wireless networks which have reciprocity
- ability to use metric methods, large flow-cut gaps for multicommodity flows in directed networks
Multi-commodity Flows

Polymatroidal network $G = (V, E)$

$k$ pairs $(s_1, t_1), \ldots, (s_k, t_k)$

Multi-commodity flow:

- $f_i$ is $s_i$-$t_i$ flow
- $f(e) = \sum_i f_i(e)$ is total flow on $e$
- flows on edges constrained by polymatroid constraints at nodes
Multi-commodity Cuts

Polymatroidal network $G=(V,E)$

$k$ pairs $(s_1, t_1), \ldots, (s_k, t_k)$

**Multicut:** set of edges that separates all pairs

**Sparsity of cut:** cost of cut/demand separated by cut

**Cost of cut:** as defined earlier via optimization
Main Results

- $\Theta(\log k)$ flow-cut gap for undir polymatroidal networks
  - throughput flow vs multicut
  - concurrent flow vs sparsest cut

- Directed graphs and symmetric demands
  - $O(\log^2 k)$ flow-cut gap for throughput flow vs multicut
  - $O(\log^3 k)$ flow-cut gap for concurrent flow vs sparsest cut

Flow-cut gap results match the known bounds for standard networks
Other Results

- $O(\sqrt{\log k})$-approximation in undir polymatroidal networks for separators (via tool from [Arora-Rao-Vazirani’04])
- Two new proofs of maxflow-mincut theorem for s-t flow in polymatroidal networks
- See paper ...
- [C-Kannan-Viswanath’12] : $O(1)$ gap for throughput flow vs multicut in planar and minor-free graphs via KPR theorem
Implications for network information theory

[Kannan-Viswanath’11] + these results imply

capacity of a class of wireless networks understood to within \( O(\log k) \) factor for \( k \)-unicast
Local vs Global Polymatroid Constraints

A more general model:

\[ G = (V, E) \text{ graph} \]

\[ f : 2^E \rightarrow \mathbb{R} \] is a polymatroid on the set of edges \( f(S) \) is the total capacity of the set of edges \( S \)

Function is global but problems become intractable

[Jegelka-Bilmes’10, Svitkina-Fleischer’09]
Technical Ideas

- Directed polymatroidal networks: a \textit{reduction via uncrossing} in the dual to standard edge-capacitated directed networks
- Undirected polymatroidal networks: \textit{dual via Lovasz-extension}
  - \textbf{sparsest cut}: round via \textit{line embeddings} inspired by [Feige-Hajiaghayi-Lee’05] on undir node-capacitated graphs
  - \textbf{multicut}: line embedding idea plus region growing [Leighton-Rao’88,Garg-Vazirani-Yannakakis’93]
Rest of talk

$O(\log k)$ upper bound on gap between max concurrent flow and min sparsity in undir polymatroidal networks
Relaxation for Sparsest Cut

Want to find edge set $E' \subseteq E$ to

\[ \text{minimize } \frac{\text{cost}(E')}{\text{dem-sep}(E')} \]

Variables:

\[ x(e) \] whether $e$ is cut or not

\[ y(i) \] whether pair $s_i t_i$ is separated or not
Relaxation for Sparsest Cut

Relaxation for standard networks:

\[ \min \sum_e c(e) \cdot x(e) \]
\[ \sum_i D_i \cdot y(i) = 1 \]
\[ \text{dist}_x(s_i, t_i) \geq y(i) \text{ for all pairs } i \]
\[ x, y \geq 0 \]

Dual of LP for max concurrent flow
Relaxation for Sparsest Cut

Relaxation for polymatroidal networks:

\[
\begin{align*}
\text{min} & \quad \text{cost of cut} \\
\sum_i D_i y(i) &= 1 \\
\text{dist}_x(s_i, t_i) &\geq y(i) \quad \text{for all pairs } i \\
x, y &\geq 0
\end{align*}
\]
Modeling cost of cut

- Each cut edge $uv$ has to be assigned to $u$ or $v$
  - Introduce variables $x(e,u)$ and $x(e,v)$ for each edge $uv$
  - Add constraint $x(e,u) + x(e,v) = x(e)$
- For a node $v$ if $S \subseteq \delta(v)$ are cut edges assigned to $v$ then cost at $v$ is $\rho_v(S)$
Relaxation for Sparsest Cut

Relaxation for polymatroidal networks:

\[
\begin{align*}
\text{min } & \text{ cost of cut} \\
\sum_i D_i y(i) & = 1 \\
x(e, u) + x(e, v) & = x(e) \text{ for each edge } uv \\
\text{dist}_x(s_i, t_i) & \geq y(i) \text{ for all pairs } i \\
x, y & \geq 0
\end{align*}
\]
Modeling cost of cut

- Each cut edge $uv$ has to be assigned to $u$ or $v$
  - Introduce variables $x(e,u)$ and $x(e,v)$ for each edge $uv$
  - Add constraint $x(e,u) + x(e,v) = x(e)$

- For a node $v$ if $S \subseteq \delta(v)$ are cut edges assigned to $v$ then cost at $v$ is $\rho_v(S)$
  - $x_v$ is the vector $(x(e_1,v),x(e_2,v),...,x(e_h,v))$ where $e_1,e_2,...,e_h$ are edges in $\delta(v)$
  - Use continuous extension $\rho_v^*(x_v)$ to model $\rho_v(S)$
Relaxation for Sparsest Cut

Relaxation for polymatroidal networks:

\[ \min \sum_v \rho^*_v(x_v) \]
\[ \sum_i D_i y(i) = 1 \]
\[ x(e,u) + x(e,v) = x(e) \text{ for each edge } uv \]
\[ \text{dist}_x(s_i, t_i) \geq y(i) \text{ for all pairs } i \]
\[ x, y \geq 0 \]
Lovasz-extension of $f$

\[ f^*(x) = \mathbf{E}_{\theta \in [0,1]}[ \ f(x^\theta) \ ] = \int_0^1 f(x^\theta) \ d\theta \]

where \( x^\theta = \{ i \mid x_i \geq \theta \} \)

Example: \( x = (0.3, 0.1, 0.7, 0.2) \)

\( x^\theta = \{1,3\} \) for \( \theta = 0.21 \) and \( x^\theta = \{3\} \) for \( \theta = 0.6 \)

\[ f^*(x) = (1-0.7) f(\emptyset) + (0.7-0.3)f(\{3\}) + (0.3-0.2) f(\{1,3\}) + (0.2-0.1) f(\{1,3,4\}) + (0.1-0) f(\{1,2,3,4\}) \]
Properties of $f^*$

- $f^*$ is convex iff $f$ is submodular
- Easy to evaluate $f^*$
- $f^*(x) = f^r(x)$ for all $x$ when $f$ is submodular
- If $f$ is monotone and $x \leq y$ then $f^*(x) \leq f^*(y)$
Relaxation for Sparsest Cut

Relaxation for polymatroidal networks:

\[
\min \sum_v \rho^*_v(x_v)
\]

\[
\sum_i D_i y(i) = 1
\]

\[
x(e,u) + x(e,v) = x(e) \quad \text{for each edge } uv
\]

\[
dist_x(s_i,t_i) \geq y(i) \quad \text{for all pairs } i
\]

\[
x, y \geq 0
\]

**Lemma:** Dual to LP for maximum concurrent flow
Rounding of Relaxation

Standard undirected networks:

- Edge capacities: round via $l_1$ embedding [Linial-London-Rabinovich’95, Aumann-Rabani’95]
- Node-capacities: round via line embedding [Feige-Hajiaghayi-Lee’05]
Line Embeddings

[Matousek-Rabinovich’01]

(V,d) metric space  \( w(uv) \) non-neg weight for each \( uv \)

\( g : V \rightarrow \mathbb{R} \) is a line embedding with average weighted distortion \( \alpha \geq 1 \) if

- \( |g(u) - g(v)| \leq d(u,v) \) for all \( u,v \) (contraction)
- \( \sum_{uv} w(uv) |g(u)-g(v)| \geq \sum_{uv} w(uv) d(uv)/\alpha \)
**Line Embeddings**

[Matousek-Rabinovich’01]

\((V,d)\) metric space  \(w(uv)\) non-neg weight for each \(uv\)

\(g : V \rightarrow \mathbb{R}\) is a line embedding with average weighted distortion \(\alpha\) if

- \(|g(u) - g(v)| \leq d(u,v)\) for all \(u,v\) (contraction)

- \(\sum_{uv} w(uv) |g(u)-g(v)| \geq \sum_{uv} w(uv) d(uv)/\alpha\)

**Theorem [Bourgain]:** Any metric space on \(n\) nodes admits line embedding with \(O(\log n)\) average weighted distortion.
Rounding Algorithm

- Solve Lovasz-extension based convex relaxation

- $x(e)$ values induce metric on $V$

- Embed metric into line with $O(\log n)$ average distortion w.r.t to weights $w(uv) = D(uv)$

- Pick the best cut $S_\theta$ among all cuts on the line
Rounding Algorithm

- Solve Lovasz-extension based convex relaxation
- \( x(e) \) values induce metric on \( V \)
- Embed metric into line with \( O(\log n) \) average distortion w.r.t to weights \( w(\text{uv}) = D(\text{uv}) \)
- Pick the best cut \( S_\theta \) among all cuts on the line
- **Remark:** Clean algorithm that generalizes edge/node/polymatroid cases since cut is defined on edges though cost is more complex
Rounding Algorithm
\( \nu(\delta(S_\theta)) \): cost of cut at \( \theta \)

**Lemma:** \( \int \nu(\delta(S_\theta)) \ d\theta \leq 2 \sum_v \rho^*_v(x_v) = 2 \text{OPT}_{\text{frac}} \)

\( D(\delta(S_\theta)) \): demand separated by \( \theta \) cut

**Lemma:** \( \int D(\delta(S_\theta)) \ d\theta \geq \sum_i D_i \text{dist}_x(s_i,t_i)/\log n \)

Therefore:

\[ \int \nu(\delta(S_\theta)) \ d\theta / \int D(\delta(S_\theta)) \ d\theta \leq O(\log n) \text{OPT}_{\text{frac}} \]
Proof of lemma

Lemma: \( \int \nu(\delta(S_\theta)) \, d\theta \leq 2 \sum_v \rho^*_v(x_v) \)

\( \nu(\delta(S_\theta)) \) is difficult to estimate exactly

Recall: \( uv \in \delta(S_\theta) \) has to be assigned to \( u \) or \( v \)
Assign according to \( x(e,u) \) and \( x(e,v) \) proportionally

\[ x(e,v) \]

\[ x'(e,v) \leq x(e,v) \]
Proof of lemma

**Lemma:** $\int \nu(\delta(S_\theta)) \, d\theta \leq 2 \sum_v \rho^*_v(x_v)$

$\nu(\delta(S_\theta))$ is difficult to estimate exactly

**Recall:** $uv \in \delta(S_\theta)$ has to be assigned to $u$ or $v$

Assign according to $x(e,u)$ and $x(e,v)$ *proportionally*

With assignment defined, estimate $\int \nu(\delta(S_\theta)) \, d\theta$ by summing over nodes
Proof of lemma

**Lemma:** \( \int \nu(\delta(S_\theta)) \, d\theta \leq 2 \sum_v \rho^*_v(x_v) \)

With assignment defined, estimate \( \int \nu(\delta(S_\theta)) \, d\theta \) by summing over nodes

\( \int \nu(\delta(S_\theta)) \, d\theta \leq 2 \sum_v \rho^*_v(x'_v) \leq 2 \sum_v \rho^*_v(x_v) \)

\( x'_v = (x'(e_1,v),...,x'(e_h,v)) \) where \( \delta(v)=\{e_1,...,e_h\} \)
Flow-Cut Gaps
Concluding Remarks

• Flow-cut gaps for polymatroidal networks match those for standard networks

Questions:

• $L_1$ embeddings characterize flow-cut gap in undirected edge-capacitated networks. What characterizes flow-cut gaps of node-capacitated and polymatroidal networks?

• What are flow-cut gaps for say planar graphs? Okamura-Seymour instances?
Thanks!
Continuous extensions of $f$

For $f : 2^N \to \mathbb{R}^+ \text{ define } g : [0,1]^N \to \mathbb{R}^+$ s.t

- for any $S \subseteq N$ want $f(S) = g(1_S)$
- given $x = (x_1, x_2, \ldots, x_n) \in [0,1]^N$ want polynomial time algorithm to evaluate $g(x)$
- for minimization want $g$ to be convex and for maximization want $g$ to be concave
Convex closure

\[ x = (x_1, x_2, \ldots, x_n) \in [0,1]^N \]

\[ \tilde{f}(x) = \min \sum_S \alpha_S f(S) \]

\[ \sum_S \alpha_S = 1 \]

\[ \sum_S \alpha_S = x_i \quad \text{for all } i \]

\[ \alpha_S \geq 0 \quad \text{for all } S \]

\[ \tilde{f}(x) \] is convex for any \( \{0,1\}^N \) function \( f \)