Multiroute Flows
&
Node-weighted Network Design

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Joint work with Alina Ene and Ali Vakilian
Survivable Network Design Problem (SNDP)

Input:
- undirected graph $G=(V,E)$
- integer requirement $r(st)$ for each pair of nodes $st$

Goal: *min-cost* subgraph $H$ of $G$ s.t $H$ contains $r(st)$ disjoint paths for each pair $st$
Steiner forest for pairs
\[ r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1 \]
SNDP Variants

Requirement

- EC-SNDP: paths are required to be edge-disjoint
- Elem-SNDP: element disjoint
- VC-SNDP: vertex/node disjoint

Cost

- edge-weights
- node-weights
## Known Approximations

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$k := \max_{st} r(st)$
$$\min \sum_e c(e) \cdot x(e)$$

$$x(\delta(A)) \geq r(st) \quad A \subseteq V, \ A \text{ separates } st$$

$$0 \leq x(e) \leq 1$$

Cut-LP for EC-SNDP

$$r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1$$
\[ \min \sum_{e} c(e) x(e) \]
\[ x(\delta(A)) \geq r(st) \quad A \subset V, \text{A separates st} \]
\[ 0 \leq x(e) \leq 1 \]

**Theorem:** [Jain] Integrality gap of Cut-LP is 2
Multi-route flows

\[ \mathcal{P}(st) = \{ p \mid p \text{ is a st path} \} \]

s-t flow, path-based defn \( f : \mathcal{P}(st) \rightarrow \mathcal{R}^+ \)

\( f(p) \) flow on path \( p \)

\[ \mathcal{P}(st, h) = \{ p = (p_1, p_2, \ldots, p_h) \mid \text{each } p_j \in \mathcal{P}(st) \text{ and the paths are edge-disjoint} \} \]

h-route s-t flow \( f : \mathcal{P}(st, h) \rightarrow \mathcal{R}^+ \)

\( f(p) \) flow on path-tuple \( p \)
Theorem: An acyclic edge s-t flow $x : E \rightarrow \mathcal{R}^+$ with value $v$ can be decomposed into a $h$-route flow iff $x(e) \leq v/h$ for all edges $e$. 

[[Kishimoto, Aggarwal-Orlin]]
Multi-route flow LP for SNDP

\[
\begin{align*}
&\text{min } \sum_{e} c(e) x(e) \\
&\sum_{p \in \mathcal{P}(st, r(st))} f(p) \geq 1 \quad \text{for all } st \\
&\sum_{p \in \mathcal{P}(st, r(st)) : e \in p} f(p) \leq x(e) \quad \text{for all } e, st \\
&0 \leq x(e)
\end{align*}
\]
Multi-route flow LP for SNDP

\[
\begin{align*}
\text{min} & \quad \sum_e c(e) x(e) \\
\sum_{p \in \mathcal{P}(st, r(st))} f(p) & \geq 1 \quad \text{for all } st \\
\sum_{p \in \mathcal{P}(st, r(st)): e \in p} f(p) & \leq x(e) \quad \text{for all } e, st \\
0 & \leq x(e)
\end{align*}
\]

Solving the LP: Separation oracle for dual is \textit{min-cost} s-t flow
Claim: Cut-LP and MRF-LP are “equivalent”

Follows from multiroute-flow theorem
Prize-collecting SNDP

Input:
- undirected graph $G=(V,E)$
- integer requirement $r(st)$ for each pair of nodes $st$
- non-negative penalty $\pi(st)$ for each pair $st$

Goal: subgraph $H$ of $G$ to minimize $\text{cost}(H) + \pi(S)$ where $S$ is set of unsatisfied pairs in $H$

All-or-nothing: $st$ satisfied if $r(st)$ disjoint paths in $H$
Prize-collecting SNDP

[Bienstock et al. 1993] Scaling trick to obtain algorithm for PC-Steiner-tree from Steiner-tree LP

[SSW 2007, NSW 2008] PC-SNDP for higher connectivity

[HKKN 2010] First constant factor for PC-SNDP in all-or-nothing model via “stronger” LP.
Prize-collecting SNDP

[BienstockGSW’93] Scaling trick to obtain algorithm for PC-Steiner-tree from Steiner-tree LP

[SSW’07, NSW’08] PC-SNDP for higher connectivity

[HKKN’10] First constant factor for PC-SNDP in all-or-nothing model via “stronger” LP.

Claim: Scaling trick of [BGSW’93] works easily for PC-SNDP via MRF-LP

“stronger” LP of [HKKN’10] equivalent to MRF-LP
MRF-LP for PC-SNDP

\begin{align*}
\min & \sum_e c(e) x(e) + \sum_{st} \pi(st) z(st) \\
\sum_{p \in \mathcal{P}(st, r(st))} f(p) & \geq 1 - z(st) \quad \text{for all } st \\
\sum_{p \in \mathcal{P}(st, r(st)): e \in p} f(p) & \leq x(e) \quad \text{for all } e, st \\
x(e) & \geq 0 \quad \text{for all } e
\end{align*}
MRF-LP for PC-SNDP

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\begin{align*}
\text{min} & \quad \sum_e c(e) x(e) + \sum_{st} \pi(st) z(st) \\
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\sum_p \in \mathcal{P}(st, r(st)) : e \in p & \quad f(p) \leq x(e) \quad \text{for all } e, st \\
x(e) & \geq 0 \quad \text{for all } e
\end{align*}
\]

**Rounding:**
- \( A = \{ st | z(st) \geq \frac{1}{2} \} \)
- Pay penalty for pairs in \( A \)
- Connect pairs *not* in \( A \)
MRF-LP for PC-SNDP

\[
\begin{align*}
\text{min} & \quad \sum_e c(e) x(e) + \sum_{st} \pi(st) z(st) \\
\sum_{p \in \mathcal{P}(st, r(st))} f(p) & \geq 1 - z(st) \quad \text{for all } st \\
\sum_{p \in \mathcal{P}(st, r(st)): e \in p} f(p) & \leq x(e) \quad \text{for all } e, st \\
x(e) & \geq 0 \quad \text{for all } e
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**Rounding:**
- \( A = \{ \text{st} \mid z(st) \geq \frac{1}{2} \} \)
- Pay penalty for pairs in \( A \)
- Connect pairs *not* in \( A \)

**Analysis:**
- Penalty for pairs in \( A \) is \( \leq 2\text{OPT} \)
- \( x'(e) = \min\{1,2x(e)\} \) is feasible for MRF-LP to connect pairs *not* in \( A \)
MRF-LP for PC-SNDP

Also extends easily to “submodular” penalty functions

Use Lovasz-extension with variables $z(st)$

([Chudak-Nagano’07] did this for Steiner tree)

Main message: $[0,1]$ variables instead of $[0,k]$ variables
Another “easy” application of multi-route flows

[Srinivasan’99] Dependent randomized rounding for multipath-routing to minimize congestion

No need for dependent rounding. [Raghavan-Thompson’87] style independent rounding works with multi-route flow decomposition

Advantages:

- Simpler and transparent
- Allows improvement via Lovasz-Local-Lemma for the short-paths case
Node-Weighted SNDP
Node-Weighted SNDP

[Klein-Ravi’95] Node-weighted Steiner tree/forest

- $O(\log n)$ approximation via “spiders”
- Reduction from Set Cover to show $\Omega(\log n)$ hardness
[Nutov’07,Nutov’09] Node-weighted SNDP

• $O(k \log n)$ approximation via generalization of spiders and augmentation framework of [Williamson etal]

• Combinatorial algorithms, not LP based
Advantages of LP-approach

[Guha-Moss-Naor-Schieber’99] LP gap of $O(\log n)$ for NW Steiner tree/forest

[Demaine-Hajia-Klein’09] LP gap of $O(1)$ for NW Steiner tree/forest in planar graphs

Via [BGSW’93] similar bounds for NW PC-ST/SF
LP for NW SNDP

Not clear! Why?
LP for NW SNDP

Not clear! Why?

EC-SNDP for a *single pair* is NP-Hard for large $k$

- $\Omega(\log n)$ hardness: easy reduction from set cover
- Consequence: Approx ratio depends on $k$

Open: approximability of single-pair for fixed $k$
MRF-LP for node weights

\[
\min \sum_v c(v) x(v) \\
\sum_{p \in \mathcal{P}(st, r(st))} f(p) \geq 1 \text{ for all } st \\
\sum_{p \in \mathcal{P}(st, r(st))} f(p) \leq x(v) \text{ for all } v, st \\
0 \leq x(v)
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x(v) \geq 0 \text{ for all } v
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Solving MRF-LP for EC-SNDP is hard

MRF-LP can be solved in poly-time for VC-SNDP!

Can solve MRF-LP for EC-SNDP within a factor of \( k \)
Integrality gap of MRF-LP

**Theorem:** Integrality gap of MRF-LP is $O(k \log n)$ for EC-SNDP and Elem-SNDP

**Theorem:** Integrality gap of MRF-LP is $O(k)$ for EC-SNDP and Elem-SNDP on planar graphs

Results extend to VC-SNDP and PC-SNDP via reductions
## Approximations for SNDP

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Approx ratios for prize-collecting problems within $O(1)$ for all probs.
Proving Integrality Gap for MRF-LP

- Augmentation framework [Williamson et al.]
- Yet another LP (Aug-LP)
- Spiders and dual-fitting for general graphs following ideas from [Guha et al. ’99, Nutov ’07, ’09]
- Primal-dual for planar graphs following [Demaine-Hajia-Klein ’09]

Some subtle technical issues
Augmentation Framework

\[ r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1 \]
Augmentation Framework

\[ r(s_1t_1) = r(s_2t_2) = 2 \text{ and } r(s_3t_3) = 1 \]

Iteration 1

Node-weighted Steiner forest problem
Augmentation Framework

\[ r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1 \]

Iteration 2

Increase connectivity by 1 for \( s_1t_1 \) and \( s_2t_2 \)

Residual graph

Covering skew-supermodular function (but arising from proper func) in residual graph
Augmentation Framework

\[ r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1 \]

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Augmentation Framework

\[ r(s_1t_1) = r(s_2t_2) = 2 \quad \text{and} \quad r(s_3t_3) = 1 \]
Augmentation Problem

$X_{i-1}$ : nodes selected in iterations 1 to $i-1$

$E_{i-1}$ : edges in $G[X_{i-1}]$, $G_i$ : residual graph $G \setminus E_{i-1}$

$f_i$ is residual covering function

$f_i(A) = 1$ if $A$ seps $st$ with $r(st) \geq i$ and $|\delta_{E_{i-1}}(A)| = i-1$

**Problem:** find min-cost set of nodes to cover $f_i$ in $G_i$

(cost of nodes in $X_{i-1}$ to 0)
Augmentation LP for phase $i$

\[
\min \sum_v c(v) x(v)
\]
\[
\sum_{v \in \Gamma(A)} x(v) \geq f_i(A) \quad \text{for all } A
\]
\[
x(v) \geq 0 \quad \text{for all } v
\]
Augmentation LP for phase i

\[
\begin{align*}
\min & \quad \sum_v c(v) x(v) \\
\sum_{v \in \mathcal{I}(S)} x(v) & \geq f_i(A) \quad \text{for all } A \\
x(v) & \geq 0 \quad \text{for all } v
\end{align*}
\]

**Theorem:** Integrality gap is \(O(\log n)\) for general graphs and \(O(1)\) for planar graphs.

If \((f,x)\) is feasible for MRF-LP then \(x\) is feasible for Aug-LP
Augmentation LP for phase i

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**Theorem:** Integrality gap is \(O(\log n)\) for general graphs and \(O(1)\) for planar graphs.

If \((f,x)\) is feasible for MRF-LP then \(x\) is feasible for Aug-LP

**Caveat:** Integrality gap is unbounded for general skew-supermodular function!
Analysis Aug-LP

- Spiders for general graphs via dual fitting
- Primal-dual for planar graphs
  - Useful lemma on node-minimal augmentation
Primal-Dual Analysis

C: minimal violated sets

[Williamson et al.] average degree of sets in \( C \) wrt to edges in an edge-minimal feasible solution is \( \leq 2 \)

**Lemma:** Number of nodes adjacent to sets in \( C \) in a node-minimal feasible solution is at most \( 4 |C| \)
Primal-Dual Analysis

**Lemma:** Number of nodes adjacent to sets in $C$ in a node-minimal feasible solution is at most $4 |C|$

By planarity average # of nodes that a set $C \in C$ is adjacent to is $O(1)$
Thank You!