Constrained Discrete-Time State-Dependent Riccati Equation Technique: A Model Predictive Control Approach

Insu Chang and Joseph Bentsman

Abstract—The continuous time state-dependent Riccati equation (SDRE) technique is extended to discrete-time under input and state constraints, yielding constrained (C) discrete-time (D) SDRE, referred to as CD-SDRE. For the latter, stability analysis and calculation of a region of attraction are carried out. The derivation of the D-SDRE under state-dependent weights is provided, and the result is compared with that of D-SDRE obtained by using the discrete-time algebraic Riccati equation (DARE) and that of discrete-time linear quadratic regulator (D-LQR). Stability of the D-SDRE feedback system is established using input-to-state stability (ISS) approach. Receding horizon strategy is used to take into account the constraints on D-SDRE controller.

I. INTRODUCTION

Over the past 40 years, several types of advanced control technologies were introduced. Unfortunately, it is true that there are still a number of unresolved problems of control applicability to real industrial systems [1]. The state-dependent Riccati equation (SDRE) technique [2] has been among the candidate techniques for addressing these problems for quite some time. The SDRE techniques are general design methods that provide a systematic and effective means of designing nonlinear controllers, observers, and filters [2]. The SDRE control techniques overcome many of the difficulties of existing methodologies such as feedback linearization, and delivers computationally efficient algorithms that are highly effective in a variety of practical applications [1].

One of the merits of the SDRE approach to nonlinear systems is to use the state-dependent coefficient (SDC) factorization that recasts a nonlinear system’s dynamics into a form resembling linear dynamics. Then, the SDRE is used to generate the feedback control law. Due to such benefits, SDRE has been applied to various nonlinear control problems: autopilot design [3], satellite attitude and orbit control [4], [5], missile guidance and control systems [6], an underactuated robot [7], a magnetically levitated ball [8], helicopters [9], a pendulum problem [10], and others.

The SDRE method was originally developed for continuous-time systems [2], [11], [12]. However, it is desirable to use a discrete-time SDRE for direct applications to real hardware embedded systems. To this end, the discrete-time version of the SDRE, called D-SDRE, has been studied as well. Dutka et al. investigated optimization of the D-SDRE [13]. In their study, two types of D-SDRE were suggested based on the discrete-time algebraic Riccati equation (DARE) and a sequential Riccati equation. A concept of MPC was used to solve the Riccati equations. Hassan used D-SDRE to design an observer-based controller [14]. D-SDRE can also be used in designing nonlinear filter systems [15], [16]. In this paper, we derive a D-SDRE feedback controller analytically by using the Hamiltonian [17]. To make the system more general, we allow weights on the performance index to be minimized to be dependent on states while previous studies assumed that they are constant or time-varying. Instead of using DARE, a generalized discrete-time Riccati equation is used. By doing so, more accurate optimization results can be expected since DARE’s assumption of steady-state conditions can lead to significant errors in a controlled system. A condition for local stability is proven by using input-to-state stability (ISS) [18], [19]. Determining a region of attraction (ROA) of the SDRE (and D-SDRE) feedback system is an open problem since an analytical solution to the SDRE is generally not known (see [20] and references therein). We suggest a way to find an ROA of a D-SDRE feedback system through the use of linear matrix inequality (LMI) methods [21], [22], [23].

The derivation and the analysis of D-SDRE without and with constraints conditions, the latter named constrained discrete-time state-dependent Riccati equation (CD-SDRE), are the main contributions of the present work. The constraint problem has been addressed through anti-windup [24], [25] and model predictive control (MPC) [26], [27], [28]. Reference [29] provides detailed information about the two techniques. In MPC, an open-loop optimal control problem is solved over a finite horizon at each sampling time, starting at the current state. Then, the control input at the current sampling time is chosen from the first element of the sequence of the calculated optimal control. MPC has been applied to an LQR under input/state constraints [30], [29], [31], [32], [33], [34], [35], [36], [37]. However, to the best of our knowledge, there are no specific results on SDRE (or D-SDRE) under input/state constraints. We suggest two algorithms for CD-SDRE: a regulation problem and a reference tracking problem. The analysis of the algorithms indicates that CD-SDRE can perform in an optimal sense in the presence of the input/state constraints. In a recent paper, Chang et al. extended the CD-SDRE to a distributed version to control multi-agent systems [38].

The rest of the paper is organized as follows. Section II briefly reviews D-LQR and MPC. In Section III, the gener-
alized D-SDRE is presented, providing derivation, stability analysis, and ROA. CD-SDRE is introduced together with numerical algorithms in Section IV. Concluding remarks are given in the last section.

II. PRELIMINARIES AND NOTATIONS

In this section, we briefly review the basic schemes of the nonlinear MPC and the D-LQR for the purpose of comparison to D-SDRE. Prior to that, we define some notations:

\[ \mathbb{N} := \{1, 2, 3, \ldots \}; \quad \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}; \quad \mathbb{Z}_{a:b} := \{z \in \mathbb{N} : z \geq a, z \leq b; a < b, a, b \in \mathbb{Z}_{\geq 0}\}; \quad \mathbb{R} := (-\infty, +\infty); \]

\[ \mathbb{R}_{\geq 0} := \{r \in \mathbb{R} : r \geq 0\}. \]

A. Discrete-Time Linear Quadratic Regulator (D-LQR)

Suppose that there is a discrete-time linear time-varying plant described by the following form

\[ x_{k+1} = A_k x_k + B_k u_k, \quad x(0) = x_0 \quad (1) \]

where \( x_k \in \mathbb{R}^n \) and \( u_k \in \mathbb{R}^m \) are the state and the control input, respectively.

The objective of D-LQR is to find the sequence of control inputs \( u_0, u_1, \ldots, u_{N-1} \) that minimize the performance index:

\[ J_0 = \frac{1}{2} \sum_{j=0}^{N-1} (x_j^T Q_j x_j + u_j^T R_j u_j) \quad (2) \]

where \( Q_j \geq 0 \) and \( R_j > 0 \) are assumed.

To this end, we use the Hamiltonian as below [39]:

\[ H_k = \frac{1}{2} (x_k^T Q_k x_k + u_k^T R_k u_k) + \lambda_{k+1}^T (A_k x_k + B_k u_k) \quad (3) \]

where \( \lambda_k \in \mathbb{R}^n \) is the Lagrange multiplier.

Then, by using the optimality condition [17], the controller can be designed as

\[ u_k = -R_k^{-1} B_k^T \lambda_{k+1} \]

\[ = - (B_k^T P_{k+1} B_k + P_k)^{-1} B_k^T P_{k+1} A_k x_k, \quad \forall k \in \mathbb{Z}_{0:N-1} \quad (4) \]

where \( P_k \) is the unique solution of the discrete-time Riccati equation at time \( k \):

\[ P_k = Q_k + A_k^T (P_{k+1} - P_{k+1} B_k)
\times (B_k^T P_{k+1} B_k + R_k)^{-1} B_k^T P_{k+1} A_k. \quad (5) \]

The detailed derivation of D-LQR is omitted here since it is straightforward and can be found in [39], [40].

**Remark 1:** If the control horizon is considered \( N \to \infty \), then (5) can be rewritten under the assumption that the state of (1) has a steady-state value:

\[ P = A_k^T (P - P B_k (B_k^T P B_k + R_k)^{-1} B_k^T P) A_k + Q_k \quad (6) \]

which is called the discrete-time algebraic Riccati equation (DARE). It is widely used in D-LQR.

B. Model Predictive Control (MPC)

MPC is the main tool in CD-SDRE technique to handle constraints on states and control inputs. Consider a discrete-time nonlinear system described by the nonlinear difference equation:

\[ x_{k+1} = f(x_k, u_k), \quad x(0) = x_0, \quad \forall k \in \mathbb{Z}_{\geq 0} \quad (7) \]

where \( f : X \times U \to X \) maps the current state \( x_k \in X \subseteq \mathbb{R}^n \) and the current control input \( u_k \in U \subseteq \mathbb{R}^m \) into the successor state \( x_{k+1} \in X \subseteq \mathbb{R}^n \).

It is assumed that the system (7) is subject to hard constraints on the state and the control input:

\[ u_k \in U, \quad x_k \in X, \quad \forall k \in \mathbb{Z}_{\geq 0} \quad (8) \]

where \( X \subseteq X, \quad U \subseteq U \), which are assumed to be closed and convex, are constraint sets of the state and the control inputs, respectively.

Then, our purpose is to find a sequence of control inputs \( \mu(\cdot) \in U \) such that the following performance index is minimized:

\[ J_0(x_0, \mu(\cdot)) := \sum_{j=0}^{N-1} \ell(x_j, u_j) + J_f(x_N) \quad (9) \]

s.t. \( x_k \in X, \quad u_k \in U \) and (7) \( \forall k \in \mathbb{Z}_{\geq 0} \)

where \( N \) is a finite horizon and \( \ell(\cdot) \) is assumed to be continuous with \( \ell(0, 0) = 0 \).

Therefore, by solving the optimal control problem [17], the optimal state and control sequence as functions of the initial state \( x_0 \) and time \( j \) can be obtained; \( \mu = [u^T(0) \ u^T(1) \ \cdots \ u^T(N-1)]^T \in \mathbb{R}^{N \times m} \) is the optimization vector. In MPC, the first element in the optimal control action \( \mu(\cdot) \) is chosen for the control input at time \( k \), i.e., \( u_k = \mu(0) \) becomes the control input signal at time \( k \), and the sequence is repeated for the next time step.

**Remark 2:** The constraints in (8) can be expressed in the following matrix form

\[ M \mu \leq W + S x_k. \quad (10) \]

Then, the minimization of (9) becomes the convex quadratic programming (QP). QP is widely used in MPC.

III. GENERALIZED DISCRETE-TIME STATE-DEPENDENT RICCATI EQUATION (D-SDRE) TECHNIQUE

In this section, we derive the D-SDRE by using the optimality condition through the use of the Hamiltonian. Then, stability conditions of the D-SDRE feedback system are provided via input-to-state stability. An ROA of a nonlinear system controlled by the D-SDRE feedback controller is investigated subsequently.
A. Derivation of the D-SDRE Feedback Controller

Consider the discrete-time nonlinear control-affine system described by the nonlinear difference equation

\[ x_{k+1} = f(x_k) + B(x_k)u_k \quad k \in \mathbb{Z}_{\geq 0} \]  

(11)

where \( x_k \in \mathbb{R}^n \) and \( u_k \in \mathbb{U} \subseteq \mathbb{R}^m \). It is assumed that \( f(0) = 0 \) and \( f(x_k) \) is continuously differentiable. In this case, the model can be rearranged through the use of the SDC factorization \[2\]:

\[ x_{k+1} = A(x_k)x_k + B(x_k)u_k \]  

(12)

We assume that \((A(x_k), B(x_k))\) is piecewise controllable for all \( x_k \in \mathbb{X} \). For this system, the D-SDRE technique finds a control input \( u_k \in \mathbb{U} \) at each time that approximately minimizes (i.e., suboptimal) the following performance index:

\[ J_0 = \frac{1}{2} \sum_{j=0}^{\infty} \left( x_j^\top Q(x_j)x_j + u_j^\top R(x_j)u_j \right) \]  

(13)

where the weights \( Q(x_j) \) and \( R(x_j) \) are assumed to be symmetric positive semi-definite and symmetric positive definite, respectively.

To find the optimal feedback controller \( u(x_k) \), the Hamiltonian defined as below is used:

\[ \mathcal{H}_k = \frac{1}{2} \left( x_k^\top Q(x_k)x_k + u_k^\top R(x_k)u_k \right) + \lambda_{k+1}^\top (f(x_k) + B(x_k)u_k) \]  

(14)

Applying the optimality condition \[40\], \[17\], we obtain the three equations:

State equation
\[ x_{k+1} = \frac{\partial \mathcal{H}_k}{\partial \lambda_{k+1}} = f(x_k) + B(x_k)u_k \]  

(15)

Costate equation
\[ \lambda_k = -\frac{\partial \mathcal{H}_k}{\partial x_k} = \bar{Q} + \bar{A}^\top \lambda_{k+1} \]  

(16)

Stationary condition
\[ 0 = -\frac{\partial \mathcal{H}_k}{\partial u_k} = B(x_k)^\top \lambda_{k+1} + R(x_k)u_k \]  

(17)

where \( \bar{Q} := Q(x_k)x_k + \frac{1}{2} x_k^\top \frac{\partial Q(x_k)}{\partial x_k} x_k + \frac{1}{2} u_k^\top \frac{\partial R(x_k)}{\partial x_k} u_k \) and \( \bar{A} := A(x_k) + \frac{\partial A(x_k)}{\partial x_k} x_k + \frac{\partial B(x_k)}{\partial x_k} u_k \).

To find the optimal solution, it is assumed that

\[ \lambda_k = P_kx_k. \]  

(18)

Substituting (18) into (17) yields

\[ u(x_k) = -R(x_k)^{-1}B(x_k)^\top \lambda_{k+1} = -R(x_k)^{-1}B(x_k)^\top P_{k+1}(A(x_k)x_k + B(x_k)u_k) \Rightarrow u(x_k) = -\left( R(x_k) + B(x_k)^\top P_{k+1}B(x_k) \right)^{-1}B(x_k)^\top \times P_{k+1}A(x_k)x_k =: -K(x_k)x_k \]  

(19)

where \( K(x_k) \in \mathbb{R}^{m \times n} \) is the suboptimal feedback control gain of the D-SDRE technique.

It should be noted that in order to obtain \( K(x_k) \), \( P_k \) and \( P_{k+1} \) are needed, which are the solutions of the generalized discrete-time Riccati equation (GD-RE) at times \( k \) and \( k+1 \), respectively. The GD-RE is derived from (15)–(19):

\[ P_k = \left( Q(x_k) + \frac{1}{2} x_k^\top \frac{\partial Q(x_k)}{\partial x_k} - \frac{1}{2} u_k^\top \frac{\partial R(x_k)}{\partial x_k} K(x_k) \right) \]  

\[ + \bar{A}^\top P_{k+1} \left( I + B(x_k)R(x_k)^{-1}B(x_k)^\top P_{k+1} \right)^{-1} \bar{A} \]  

(20)

Remark 3: The algebraic Riccati equation (ARE) is used in LQR problems. In \[2\] and many other studies on SDRE, the ARE has been commonly used. Likewise, the DARE in \[5\] can be used for D-SDRE (Algorithm 1 in \[13\]). In this case, there is an assumption that \( A(x_k) = \frac{\partial}{\partial x} f(x_k) + B(x_k)u_k \) \( \forall x_k \in \mathbb{X} \). However, it is not satisfied in general. Therefore, the feedback controller \( u_k \) may not work properly in an optimal sense unless \( \frac{\partial A(x_k)}{\partial x_k} x_k + \frac{\partial B(x_k)}{\partial x_k} u_k \neq 0 \) \( \forall x_k \in \mathbb{X} \).

Remark 4: In this paper, \( Q \) and \( R \) in (13) are assumed to be dependent on the state \( x_k \), i.e., \( Q = Q(x_k) \) and \( R = R(x_k) \). For simplicity, \( Q \) and \( R \) can be considered to be independent on \( x_k \). Then, \( Q = \bar{Q} \) in (16), and (20) becomes the same formula as that in Algorithm 2 in \[13\]. However, since \( Q \) and \( R \) affect the performance of an optimal control problem such as D-SDRE, it is more desirable to use state-dependent matrices rather than constant ones.

Remark 5: In (18), we assumed that \( P_k \) is not explicitly dependent on the current state \( x_k \). Therefore, (20) can be solved sequentially with the initial condition \( P_0 = 0 \) with a proper dimension.

Remark 6: The reason of the suboptimal controller \( u(x_k) = -K(x_k)x_k \) instead of the optimal controller is because of the non-uniqueness of \( f(x_k) = A(x_k)x_k \). That is, there are \( A_1(x_k) \) and \( A_2(x_k) \) such that \( A_1(x_k)x_k = A_2(x_k) = f(x_k) \). There are numerous ways to choose \( A(x_k) \). Therefore, the choice of \( A(x_k) \) can affect the performance of the controller.

B. Stability Analysis of D-SDRE

It should be noted that the D-SDRE feedback controller is locally stabilizing the discrete-time nonlinear difference equation (11) or (12). In this part, we investigate the stability of the D-SDRE controller. Prior to that, we use the following function classes. A function \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is said to be of class \( \mathcal{K} \) if it is continuous, strictly increasing, and \( \gamma(0) = 0 \). If \( \gamma \) is unbounded, it is said to be of class \( \mathcal{K}_{\infty} \). A function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is said to be of class \( \mathcal{KL} \) if \( \beta(\cdot, k) \) is of class \( \mathcal{K} \) for each fixed \( k \geq 0 \) and \( \beta(\xi, \cdot) \) is decreasing to zero as \( k \to \infty \) for each fixed \( k \geq 0 \).

We introduce the concept of input-to-state stability \[18\], \[19\].

Definition 1: The discrete-time nonlinear control-affine system (12) is said to be input-to-state stable (ISS) if there exist \( \beta \in \mathcal{KL}, \gamma \in \mathcal{K}, \) and constant \( \eta_1, \eta_2 \in \mathbb{R}_{\geq 0} \) such that

\[ \|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|u\|_{\infty}) \quad \forall k \in \mathbb{Z}_{\geq 0} \]  

(21)
for all \( x_0 \in \mathbb{X} \) and \( u_k \in U \) satisfying that \( \|x_0\| < \eta_1 \) and \( \|u\|_{L_\infty} < \eta_2 \).

We also use the following definitions [19]:

**Definition 2:** A continuous function \( V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) is said to be an ISS-Lyapunov function for (12) if the following hold:

1. There exist \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) such that
   \[
   \alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n.
   \] (22)

2. There exist \( \alpha_3 \in \mathcal{K}_\infty \) and \( \sigma \in \mathcal{K} \) such that
   \[
   V(f(\xi, \mu)) - V(\xi) \leq -\alpha_3(\|\xi\|) + \sigma(\|\mu\|)
   \] (23)

   for all \( \xi \in \mathbb{R}^n \) and \( \mu \in \mathbb{R}^m \).

**Theorem 1:** The discrete-time nonlinear system (12) controlled by the D-SDRE technique (19) is locally ISS in the ROA of the system.

**Proof:** By substituting (19) into (12), we obtain the closed-loop form of the system:

\[
x_{k+1} = (A(x_k) - B(x_k) \left( R(x_k) + B(x_k)^T P_{k+1} B(x_k) \right)^{-1} \times B(x_k)^T P_{k+1} A(x_k) ) x_k =: A_{cl}(x_k) x_k.
\] (24)

Then, we can derive the following equation by using the direct substitution into (24):

\[
x_{k+1} = \prod_{j=0}^{k} A_{cl}(x_j) x_0.
\] (25)

By assumption in (12), \( (A(k), B(k)) \) is piecewise controllable for \( x_k \in \mathcal{X} \), which implies that the system is stabilizable. Then, for \( \zeta < 1 \), there exist \( c \geq 0 \) and \( 0 \leq \sigma < \zeta \) such that

\[
\|A_{cl}(x_k)\| \leq c \sigma_k \leq c \sigma_{\max}
\] (26)

where \( \sigma_{\max} = \max_k \sigma_k \).

We need to find an ISS-Lyapunov function of (12). Given \( D = D^T > 0 \), we can find \( M_k = M_k^T > 0 \) for each \( k \) from the following equation:

\[
A_{cl}^T(x_k) M_k A_{cl}(x_k) - M_{k-1} = -D.
\] (27)

Consider the following ISS-Lyapunov function candidate:

\[
V(x_k) = x_k^T M_{k-1} x_k
\] (28)

It shows that \( V(x_k) \geq \lambda_{\min} \|x_k\|^2 =: \alpha_1(\|x_k\|) \) and \( V(x_k) \leq \lambda_{\max} \|x_k\|^2 =: \alpha_2(\|x_k\|) \) where

\[
\lambda_{\min} = \min_k \lambda_{\min}(M_k), \quad \lambda_{\max} = \max_k \lambda_{\max}(M_k).
\] (29)

Then, we can find \( \alpha_3(\|x_k\|) \) in Definition 2 from (12), (24)–(29):

\[
V(x_{k+1}) - V(x_k) = x_{k+1}^T M_{k-1} x_{k+1} - x_k^T M_{k-1} x_k
\]

\[
= x_k^T A_{cl}^T M_k A_{cl} x_k - x_k^T M_{k-1} x_k
\]

\[
= -x_k^T D x_k \leq -\lambda_{\min}(D) \|x_k\|^2 =: -\alpha_3(\|x_k\|),
\] (30)

which implies that \( V \) in (28) is an ISS-Lyapunov function and therefore, the system (12) controlled by (19) is locally ISS.

**C. Estimates of Region of Attraction (ROA) of D-SDRE**

We consider a discrete-time nonlinear system controlled by the D-SDRE technique. We have shown that given the nonlinear system (11), under the assumption of an autonomous nonlinear equation, it can be rewritten in the form of (12) by using the SDC factorization. Moreover, by (19), the closed-loop system controlled by the D-SDRE feedback controller was obtained in (24). However, it should be emphasized that it is difficult to express \( A_{cl}(x_k) \) in an analytic form due to the difficulty of an analytic expression of \( P_k \) and \( P_{k+1} \). Therefore, we should approach the problem in a different way: one can view this problem as a robust stability problem by assuming that \( A_{cl}(x_k) \) is not precisely known, but it stays in \( \mathcal{G} \), a convex, closed, and bounded domain (polytope) [23], i.e.,

\[
\mathcal{G} = \left\{ A_{cl}(\alpha) : A_{cl}(\alpha) = \sum_{i=1}^{L} \alpha_i A_{cl,i}; \sum_{i=1}^{L} \alpha_i = 1; \alpha_i \geq 0 \right\}
\] (31)

where \( A_{cl,i} \) are the vertices of the polytope \( \mathcal{G} \).

Then, we have useful lemmas to find the ROA of the D-SDRE feedback system.

**Lemma 2:** Suppose \( A_{cl}(x_k) \) has uncertainties but belongs to \( \mathcal{G} \) in (31). Then, (24) is robustly stable in \( \mathcal{G} \) if there exists \( P = P^T > 0 \) such that

\[
A_{cl,i}^T P A_{cl,i} - P < -\rho I
\] (32)

for all \( i = 1, 2, \cdots, L \) and \( \rho > 0 \).

**Proof:** See [22] and [19].

**Lemma 3:** Suppose \( A_{cl}(x_k) \) has uncertainties but belongs to \( \mathcal{G} \) in (31). Then, (24) is robustly stable in \( \mathcal{G} \) if there exist \( P_i = P_i^T > 0 \) and \( G \) such that

\[
\begin{bmatrix}
P_i & A_{cl,i}^T G^T \\
G A_{cl,i} & G + G^T - P_i
\end{bmatrix} > \rho I
\] (33)

for all \( i = 1, 2, \cdots, L \) and \( \rho > 0 \).

**Proof:** See [22].

Note that Lemmas 2 and 3 can be easily established by standard LMI solvers since \( A_{cl,i} \), the vertices of the polytope \( \mathcal{G} \), are linear and so is (33) on \( P_i \). Therefore, the ROA of D-SDRE feedback control system

\[
\mathcal{X} = \left\{ x \in \mathcal{X} : A_{cl}(x) \in \mathcal{G} \text{ in (31)} \right\}.
\] (34)
So far, we derived the D-SDRE feedback controller and proved the stability condition in an ROA which is obtained numerically via LMIs. Notice that we assumed that there are no constraints on the states or the control inputs. In the next section, we will show how to handle the constraints on the D-SDRE technique.

IV. CONstrained Discrete-Time State-Dependent Riccati Equation (CD-SDRE) Technique

In this section, we consider the D-SDRE technique under the constraints on the states and the control inputs. We investigate the problem in two different cases: regulation and reference tracking problems.

A. Regulation Problem of CD-SDRE

In order to provide an algorithm for the D-SDRE with constraints on the states and the control inputs, we define the problem first. Consider the discrete-time nonlinear control-affine system (12), described by using the nonlinear difference equation. Then, we want to design the D-SDRE state feedback controller \( u(x_k) \) as in (19) such that the performance index is minimized:

\[
J(x_0, \mu(\cdot)) := \sum_{k=0}^{k_f-1} x_k^T Q(x_k) x_k + u_k^T R(x_k) u_k \tag{35}
\]

s.t. \( x_{k+1} = f(x_k) + B(x_k) u_k, \quad x(0) = x_0 \)
\( x_k \in \mathcal{X}, \quad u_k \in \mathcal{U} \quad \forall k \in \mathbb{Z}_{\geq 0} \)

where \( \mathcal{X} \) and \( \mathcal{U} \) are closed, bounded, and convex, and contain the origins in their interiors.

Then, the following algorithm suggests the CD-SDRE technique.

**Algorithm 1: Regulation Problem of CD-SDRE**

Define \( \mathcal{X} \subseteq \mathcal{X} \subseteq X \subseteq \mathbb{R}^n \) and \( \mathcal{U} \subseteq U \subseteq \mathbb{R}^m \).

For \( k = 0 : 1 : k_f - 1 \)

Solve (20) sequentially to find \( u(x_k) = -K(x_k)x_k \).

If \( u(x_k) \in \mathcal{U} \) and \( x_k \in \mathcal{X} \) (unconstrained case)
\( u_k = -K(x_k)x_k \).

Else (constrained case)
Define \( N \).

For \( j = k : 1 : N + k \)

Solve (35) to obtain \( \mu(\cdot) \).
\( u_k = \mu(0) \).

End
End

Notice that Algorithm 1 is for the regulation problem where the reference is assumed to be constant. It can be extended to the case where the reference is time-varying; Algorithm 2 introduced in the next shows the reference tracking problem with constraints on states and control inputs.

B. Reference Tracking Problem of CD-SDRE

Consider the same nonlinear discrete-time system in (12). In this case, we want to design CD-SDRE optimal feedback controller \( u_k \) such that \( x_k \) tracks a time-varying reference \( r_k \) in an optimal sense. Then, the performance index (35) to be minimized is modified to the following form:

\[
J(x_0, \mu(\cdot)) := \sum_{k=0}^{k_f-1} (x_k - r_k)^T Q(x_k)(x_k - r_k) + u_k^T R(x_k) u_k \tag{36}
\]

s.t. \( x_{k+1} = f(x_k) + B(x_k) u_k, \quad x(0) = x_0 \)
\( x_k \in \mathcal{X}, \quad u_k \in \mathcal{U} \quad \forall k \in \mathbb{Z}_{\geq 0} \)
\( \delta u_k \in \delta \mathcal{U} \)

where \( \delta u \) denotes the increment of the control input at time \( k \), i.e., \( \delta u_k = u_k - u_{k-1} \).

Then, the following algorithm suggests the CD-SDRE technique for the reference tracking.

**Algorithm 2: Reference Tracking Problem of CD-SDRE**

Define \( \mathcal{X} \subseteq \mathcal{X} \subseteq X \subseteq \mathbb{R}^n \) and \( \mathcal{U} \subseteq U \subseteq \mathbb{R}^m \).

For \( k = 0 : 1 : k_f - 1 \)

Solve (20) sequentially to find \( u(x_k) = -K(x_k)x_k \).

If \( u(x_k) \in \mathcal{U} \) and \( x_k \in \mathcal{X} \) (unconstrained case)
\( u_k = -K(x_k)x_k \).

Else (constrained case)
Define \( N \).

For \( j = k : 1 : N + k \)

Solve (36) and obtain \( \mu(\cdot) \).
\( u_k = \mu(0) \).

End
End

Remark 7: As mentioned in Remark 2, the constraints can be expressed in a matrix form to use the quadratic programming (QP). That is, the constraints in (35) and (36) can be expressed by using the matrix formula to use QP. Readers are referred to [27] and references therein for more detailed information.

We have discussed constraint problems of the D-SDRE controller. Through the use of MPC, the D-SDRE is able to generate suboptimal control signals in the presence of constraints on inputs/states.

V. CONCLUDING REMARKS

In this paper, we investigated the discrete-time nonlinear system controlled by the D-SDRE technique. The D-SDRE controller with the state-dependent weights was derived by directly using the Hamiltonian and the optimality conditions. The stability analysis was done by means of input-to-state stability. We suggested a method to estimate an ROA where
the D-SDRE feedback system is ISS by using LMI methods. Finally, we suggested algorithms of the CD-SDRE techniques for the regulation and the reference tracking problems when constraints must be considered.

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REFERENCES


