Project Report: Regularization Methods for Image Deblurring

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Abstract Restoring an image has become increasingly important in different scientific applications, such as astronomy, satellite surveillance, medical imaging, and others. Being able to deblur these images, to remove the noise from them, can help scientists gain a better insight into their data. In this project we study two different numerical methods (i.e., truncated singular value decomposition, Tikhonov regularization) used in image processing to restore different images, and also show why using the naive approach in this case doesn’t work. Numerical experiments will be presented.

Key words. image processing, filtering, Matlab, truncated singular value decomposition, Tikhonov regularization

1 Introduction

Images both of a personal nature or of a scientific importance, they all carry information. However, as of any other form of data, the information within each picture can be affected by errors. These errors can come from various sources: motion while the picture is taken, unfocused lens, light, or in the case of astronomical images, from atmospheric turbulence. These errors are often responsible, along with the noise, for the blurred and unclear images.

Being able to restore an image is important, because it may reveal different details that are hidden from the human eye: a far away galaxy that appears as a star, the face of a criminal, or an X-ray of a broken leg.

Mathematically, images can be represented as two or three dimensional arrays, in which each element represents an intensity in a gray or color scale. The blurring image can then be modeled by the linear system:

\[ Ax = b \]  

(1.1)

where \( A \) is the blurring matrix, \( x \) is the exact, sharp image, and \( b \) is the blurred image. Then, the problem of reconstructing the true image reduces to solving 1.1 for \( x \). [3, 7] Unfortunately, this is often not easy. Choosing the right algorithm that will either do a complete recovery or minimize the existing errors, proves to be one of the big challenges.
In this report we study two of the most used numerical linear algebra methods for reconstruction, based on the filtering factor used in the regularization of the images. In Section 2 we discuss the truncated singular value decomposition and the Tikhonov regularization. In Section 3 we present an analysis and comparison of the two methods, and we discuss their efficiency. Then in Section 4 we present our observations and conclusions.

2 Truncated Singular Value Decomposition and Tikhonov Regularization

The image in Figure 2 has been computed by directly using the inverse: $x = A^{-1}b$. As we can see, the result image we have gotten is even less similar to the original one shown in Figure 1. Instead of smoothing over the errors, the naive approach is amplifying them. This is explained by the fact that certain types of errors have been omitted; the noise in the image has also been inverted.

Singular value decomposition (SVD) is an important factorization tool used in the study of linear systems of equations. [8]

For a square n-by-n matrix $A$, it is defined as:

$$A = U \Sigma V^T$$

where $U$ and $V$ are orthogonal matrices, $\Sigma$ is a diagonal matrix with

$$\sigma_{ij} = \begin{cases} 0, & i \neq j \\ \sigma_i \geq 0, & i = j \end{cases}$$

and $\sigma_i$ are singular values of $A$, ordered decreasingly in the matrix $\Sigma$. The columns $u_i$ of $U$ and $v_i$ of $V$ are the left, and right, respectively, singular vectors.

Piecewise, the SVD factorization of $A$ can be rewritten as:

$$A = \sum_{i=j}^{n} u_i \sigma_i v_i^T$$
The solution of 1.1 for $x$ is analytically $A^{-1}b$. Using singular value decomposition also for the inverse of the blurring matrix $A$:

$$A^{-1} = V\Sigma^{-1}U^T = \sum_{i=1}^{n} \frac{1}{\sigma_i} u_i^T v_i$$

we then have the following identity:

$$x = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i$$

since $x = A^{-1}b$.

For a typical deblurring image problem, matrix $A$ has many values very close to zero, implying the singular values are also very close to zero [2, 7]. In this context, the deblurring matrix $A$ behaves like a singular matrix. It has a very large condition number and an error very sensitive to perturbation.

However, the conditioning can be improved by considering throwing out small singular values; this can be done by removing the singular values from the decomposition of the deblurring matrix that are zero or close to zero. Since the singular values are ordered decreasingly in the $\Sigma$ matrix, the process of removing the smallest ones is straightforward: it can be done by truncation. For this reason, the method is called, of course, truncated singular value decomposition (TSVD). TSVD changes a problem from ill-posed to well-posed.

In Matlab, one can find the true image with the following command:

```matlab
[U,S,V] = svd(A);
x=V(:,1:k)*pinv(S(1:k,1:k))*U(:,1:k)'*b
```

where $A$ is the blurring matrix, $b$ is the blurred image (with or without the noise, depending on the problem) and $k$ is the parameter that specifies the number of the last singular value...
that is not close to zero. An estimate for $k$ can be deduced by plotting $\Sigma$, the singular values matrix.

One way to stabilize a numerical algorithm is through regularization; and one example is the truncated singular value decomposition presented above. Another example is the Tikhonov regularization, presented below.

Regularization avoids highly oscillatory eigenvectors, which are responsible for amplifying the noise and errors in the blurred image, $b$ [2].

The general solution form for finding the true image using regularization is the following [7]:

$$x_{filtered} = \sum_{i=1}^{N} \Phi_i \frac{u_i^T b}{\sigma_i} v_i$$

For the TSVD case, $\Phi_i$ is the filter factor:

$$\Phi_i = \begin{cases} 1, & i = 1, \ldots, k \\ 0, & i = k + 1, \ldots, N \end{cases}$$

where $N$ is the dimension of square matrix $A$. $k$ is called the truncation parameter.

The Tikhonov filter factor is defined [3, 4, 7] as:

$$\Phi_i = \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2}$$

$\alpha$ is called a regularization parameter that is chosen from the interval $[\sigma_{\text{min}}, \sigma_{\text{max}}]$ and which has the same role as the parameter $k$ from the TSVD method.

The process of choosing the right $\alpha$ and $k$ requires use of "automatic" methods; however, they can be chosen "randomly" and the "best random" value will give the "best picture", which will most closely resemble the initial picture visually.

### 3 Comparison of the two methods

For a random interval of singular values we plot in Figure 3 the filter factor $\sigma_i$ versus the singular values of the TSVD and Tikhonov decompositions. The filter parameter $\alpha$ was chosen to be 15. From this graph of an abstract problem, we see that the Tikhonov function is much, much smoother, looking more as an exponential function.

The authors of [7], whom we tried to follow in our study of the two regularization method have made available a series of efficient codes for computing the solutions that approximate the best the true image, the filter parameters $\alpha$ and $k$, and the blurring matrix. The codes can be downloaded from [10].

In Figure 4 we show the solutions of using the truncated singular value decomposition and Tikhonov regularization methods applied to the blurred image from Figure 1. The Tikhonov produces slightly better results.
Figure 3: Comparing the Tikhonov method to TSVD, for a filter parameter equal to 15, in a randomly selected singular values interval.

Figure 4: Solutions of Tikhonov and TSVD reconstruction methods.
4 Conclusion and Observations

From restoring vacation photos to analyzing astronomical data, image reconstruction is a useful technique. In this study project we have considered two of the most used methods in the field: the truncated singular value decomposition and Tikhonov regularization methods.

We have shown that the naive approach of using a direct inverse does not usually work for image recovering, due to the large and ill-conditioned deblurring matrices, that are the equivalent of a coeffiecient matrix in the linear system of equations $Ax = b$. The reason is that the noise and errors in the right hand side vectors are amplified, deforming the images even more.

Instead, an approach that uses singular value decomposition can be used in solving the system 1.1. Because of the ill-conditioning problem, which include zero or almost zero singular values, SVD cannot be used as a stand alone method. Further work requires removing these small values from the matrix decomposition. Regularization methods as the ones studied in this report, can help achieve this.

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References


