The paper discusses some of the game theoretic results that are closely related to the robust control design. Robust control theory is a method to measure the performance changes of a control system with changing system parameters. The controller is called robust, if it would achieve good performance and stability in the presence of bounded modeling errors. On the other hand, the theory of Games is concerned with decision making in the presence of other controllers/players, each of which has their own payoff function and own set of information, which need not be necessary independent and identical. The paper exploits the direct connection between the two fields to come up with a common theoretical background for solving either problem.

**Game Theory and Saddle Point:**

Let \( \phi(u,w) : U \times W \rightarrow R \) be a function and consider the problems:

\[
\inf_u \sup_w \phi(u,w) = \gamma^* \quad \text{and} \quad \sup_w \inf_u \phi(u,w) = \gamma^* \quad (1)
\]

Also, we have

\[
\gamma^* \geq \gamma^* \quad (2)
\]

A pair \((\bar{u}, \bar{w}) \in U \times W\) is called the saddle point if

\[
\phi(\bar{u}, w) \leq \phi(\bar{u}, \bar{w}) \leq \phi(u, \bar{w}) \quad \forall u \in U, w \in W \quad (3)
\]

**Theorem:** Let \(U, W\) be nonempty. A pair \((\bar{u}, \bar{w})\) solves (3) iff

1) \(\bar{u}\) is min max optimal.
2) \(\bar{w}\) is max min optimal.
3) The duality gap \(\gamma^* - \gamma^*\) is zero.

**Standard \(H^\infty\) Problem:**

Consider the following linear control system:

\[
\begin{align*}
\dot{x} &= Ax + Bu + \Gamma w \\
x(0) &= 0 \\
z &= Cx + Du
\end{align*} \quad (4)
\]

with the objective to minimize the total output energy in presence of the external bounded disturbance, or,

\[
\inf_u \sup_w \phi(u,w) = \inf_u \sup_w \|z\|_2^2 \quad (6)
\]

subject to:

\[
\|w\|_2^2 = \int_0^\infty w'(t)w(t)dt \leq 1 \quad (7)
\]

The paper doesn’t discuss on how to solve this problem. However, I thought of coming up with a derivation, as we will be following a similar approach throughout this article.

Let us try to solve the above problem (assuming a saddle point solution exists):
If solution to above Riccati equation exists, then the feedback control is given by:

\[ J(u, w) = \int_0^\infty \left( z'(t)z(t) - \gamma^2(w'(t)w(t) - 1) \right) dt \]  

where, \( \gamma^2 \) is the Lagrange multiplier.

**Assumptions:** \( C^T C := Q, C^T D = 0 \) and \( D^T D = I \). Under these set of assumptions, (8) can be re-written as:

\[ J(u, w) = \int_0^\infty \left( x^T Q x + u^T u - \gamma^2 (w^T w - 1) \right) dt \]  

The Hamiltonian \( H \) for this problem is defined as:

\[ H(t, x, u, w, p) := \langle p, \dot{x} \rangle - L(t, x, u, w) \]

\[ H(t, x, u, w, p) := \langle p, Ax + Bu + \Gamma w \rangle - \left( x^T Q x + u^T u - \gamma^2 (w^T w - 1) \right) \]  

where, \( p \) is the co-vector.

From the Hamiltonian Canonical Equations:

\[ \dot{p}^* = -H^*_x \]

\[ \Rightarrow \dot{p}^* = -A^T p^* + 2Qx^* \]  

And

\[ \dot{x}^* = H^*_p \]

\[ \Rightarrow \dot{x}^* = Ax^* + Bu^* + \Gamma w^* \]

Also, from Hamiltonian-maximization condition (assuming Saddle point)

\[ u^* = \arg \max_{u \in U} H(t, x^*, u, w^*, p^*) \]

\[ \Rightarrow u^* = \frac{1}{2} B^T p^* \]  

\[ w^* = \arg \min_{w \in W} H(t, x^*, u^*, w, p^*) \]

And,

\[ \Rightarrow w^* = -\frac{1}{2\gamma^2} \Gamma^T p^* \]  

From (4), (11), (12), (13) and (14),

\[ \begin{bmatrix} \dot{x}^* \\ \dot{p}^* \end{bmatrix} = \begin{bmatrix} A & \frac{1}{2} \left( B B^T - \frac{1}{\gamma^2} \Gamma \Gamma^T \right) \\ 2Q & -A^T \end{bmatrix} \begin{bmatrix} x^* \\ p^* \end{bmatrix} \]  

Let \( \mu(t, t_i) \) be the state transition matrix, where,

\[ \mu(t, t_i) := \begin{bmatrix} \mu_{11}(t, t_i) & \mu_{12}(t, t_i) \\ \mu_{21}(t, t_i) & \mu_{22}(t, t_i) \end{bmatrix} \]  

Hence, from (15) and (16),

\[ \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} = \begin{bmatrix} \mu_{11}(t, t_f) & \mu_{12}(t, t_f) \\ \mu_{21}(t, t_f) & \mu_{22}(t, t_f) \end{bmatrix} \begin{bmatrix} x^*(t_f) \\ p^*(t_f) \end{bmatrix} \]  

For infinite horizon,

\[ \lim_{t \to \infty} p^*(t_f) = 0 \]

\[ \Rightarrow p^*(t_f) = \mu_{21} \mu_{11}^{-1} x^*(t_f) := -2P x^*(t) \]  

Substituting (18) into the system equation (4), we obtain the following Riccati Equation in \( P \):

\[ -\dot{P} = PA + A^T P - P(B B^T - \frac{1}{\gamma^2} \Gamma \Gamma^T) P + Q \]

\[ \lim_{t \to \infty} P(T, \gamma) = 0 \]  

which is also known to arise in **LQ Game Theory Problems**.

If solution to above Riccati equation exists, then the feedback control is given by:

\[ u^*(t) = -B^T P(t, \gamma^*) x(t) \]
Assume that there exists a pair \((\overline{u}, \overline{w}) \in U \times W\) for some \(\varepsilon > 0\) such that,
\[
-\varepsilon + \phi(\overline{u}, w) \leq \phi(\overline{u}, \overline{w}) \leq \varepsilon + \phi(u, \overline{w}) \quad \forall u \in U, w \in W \tag{21}
\]
and using the fact,
\[
\gamma^* = \inf_u \sup_w \phi(u, w) \leq \sup_w \phi(\overline{u}, w) \tag{22}
\]
yields,
\[
\gamma_* \leq \gamma^* \leq \gamma_* + 2\varepsilon \tag{23}
\]

**Decentralized Multiobjective H\(^\infty\) Control:**
Consider the decentralized control system, depicted in the figure below:

![Control system diagram](image)

We further assume that \(D_i^T H_i = 0\) and define \(Q_i = H_i^T H_i\). Also, define the closed-loop transfer function by:
\[
T_{zc} := \begin{bmatrix} T_{z_1w} \\ T_{z_2w} \end{bmatrix}
\]

The objective is to design feedbacks \(u_1(y_1)\) and \(u_2(y_2)\) such that
\[
\begin{align*}
\inf_{u_1, u_2} & \sup_{w \neq 0} \frac{\lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2}{\|w\|^2} \\
& \leq \delta_1 \\
& \leq \delta_2
\end{align*}
\]

If both controllers act as a team, they agree on some relative weights \(\lambda_1, \lambda_2, 0 \leq \lambda, \lambda_1 + \lambda_2 = 1\), and we have the problem,
\[
\inf_{u_1, u_2} \sup_{w \neq 0} \frac{\lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2}{\|w\|^2} \tag{25}
\]

Let \(K_1 = (F_1, G_1, L_1, M_1)\) and \(K_2 = (F_2, G_2, L_2, M_2)\) be the required feedback controllers. The game can be solved by approaching in a similar way as shown earlier:

To solve this problem, we consider the associated game:
\[
\begin{align*}
\sup_{\alpha} & \lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2 - \gamma^2 \|w\|^2, \quad K_1, K_2 \text{ fixed} \tag{26} \\
\inf_{\alpha} & \lambda_1 \|z_1\|^2 + \lambda_2 \|z_2\|^2, \quad w \text{ fixed} \tag{27}
\end{align*}
\]

(26) is a standard LQ problem and the associated Riccati Equation:
\[
0 = PA + A^T P + \tilde{Q} + \frac{1}{\gamma} P \tilde{D}^T \tilde{P} P \tag{29}
\]

where,
\[ A = \begin{bmatrix} A + B_1 M_1 C_1 + B_2 M_2 C_2 & B_1 L_1 & B_2 L_2 \\ G_1 C_1 & F_1 & 0 \\ G_2 C_2 & 0 & F_2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} \lambda_1 Q_1 + \lambda_2 Q_2 + C_1^T M_1^T M_1 C_1 + C_2^T M_2^T M_2 C_2 & C_1^T M_1^T \end{bmatrix} \]

**Paper II Review: $H^\infty$ Optimal Control for Singly Perturbed Systems**

**Part 1: Perfect State Measurements**

Zigang Pan & Tamer Basar

The paper shows that as the singular perturbation parameter approaches zero, the optimal disturbance attenuation level for the full-order system under a quadratic performance index converges to a value that is bounded above by the maximum of the optimal disturbance attenuation levels for the slow and fast subsystems under appropriate "slow" and "fast" quadratic cost functions. The paper also shows that under certain technical conditions, the “slow” controller can be used reliably for sufficiently small $\varepsilon$, and the fast dynamics can be completely ignored.

Singly perturbed problems are generally characterized by dynamics operating on multiple scales. A **singly perturbed system** is a system containing a small parameter $\varepsilon$ that cannot be approximated by setting the parameter value to zero.

**Problem Formulation:**

Consider a “singly perturbed” system, whose dynamics is given by,

\[
\dot{x}_1 = A_{11}(\varepsilon)x_1 + A_{12}(\varepsilon)x_2 + B_1(\varepsilon)w; \quad x_1(0) = 0, \quad x_1 \in \mathbb{R}^n
\]

\[
\dot{x}_2 = A_{21}(\varepsilon)x_1 + A_{22}(\varepsilon)x_2 + B_2(\varepsilon)u + D(\varepsilon)w; \quad x_2(0) = 0, \quad x_2 \in \mathbb{R}^{n_2}
\]

where, $x' = (x_1, x_2)$ is the $n$-dimensional vector, $w$ is the disturbance signal and $u$ is the feedback control input, given by:

\[
u(t) = \mu(t, x_{[0,t]})
\]

where, $\mu$ is piecewise continuous in $t$, Lipschitz continuous in $x$, further satisfying **Causality** conditions.

The associated standard quadratic performance index for this system is:

\[
L(u, w) = \|x(t)\|^2_{Q_f} + \int_0^t \left(\|x(t)\|^2_{Q_2(t)} + \|\mu(t)\|^2\right) \, dt
\]

\[
\equiv \|x(t)\|^2_{Q_f} + \|x\|^2_{Q_2} + \|\mu\|^2; \quad Q_f \geq 0 \quad Q_2 \geq 0
\]

where, $Q_f$ is dependent on $\varepsilon > 0$.

The associated $H^\infty$ optimal control problem is to minimize,

\[
\sup_{\text{we} H_*} \frac{L(\mu(t, x_{[0,t]}), w)^{1/2}}{\|w\|} = \sup_{\text{we} H_*} \frac{J(\mu, w)^{1/2}}{\|w\|}
\]

Denote the infimum, $\gamma^*(\varepsilon)$ as:

\[
\gamma^*(\varepsilon) = \inf_{\mu \in M} \sup_{\text{we} H_*} \frac{J(\mu, w)^{1/2}}{\|w\|}
\]

Clearly, for each fixed $\varepsilon > 0$, one can associate a soft-constrained LQDG (Linear Quadratic Differential Game) with the worst-case design problem, with cost-function:

\[
L_{\gamma}(u, w) = L(u, w) - \gamma^2 \|w\|^2
\]

As shown in the previous summary, for every $\gamma > \gamma^*(\varepsilon)$, solving (35) is equivalent to solving the associated Riccati equation since this differential-game admits a saddle-point solution, with the saddle-point controller $\mu^*$ being a linear feedback-law. However, due to numerical stiffness, the computation of $\gamma^*(\varepsilon)$ and a corresponding $H^\infty$ optimal or suboptimal controller for small
values of $\epsilon$ is seriously difficult. The contribution of this paper is to come up with an $\epsilon$-independent small-order problem, the solution to which determines $\gamma^\ast(\epsilon)$.

**Assumptions:**

A1.  

$Q_f = \begin{bmatrix} Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \epsilon Q_{22} \end{bmatrix}$,  

$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$

A2.  

$A_i(t), Q_i(t), B_i(t), D_i(t) (i, j = 1, 2)$: continuously differentiable in $t \geq 0$

A3.  

$A_{22}(t)$ and $Q_{22}(t)$ : invertible for all $t \in [0, t_f]$

Let us introduce further notations, $A_i(t)$, $B_i(t)$ and $D_i(t)$ such that,

$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix},$  

$A_\epsilon(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ (1/\epsilon)A_{21}(t) & (1/\epsilon)A_{22}(t) \end{bmatrix}$  

$B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix},$  

$B_\epsilon(t) = \begin{bmatrix} B_1(t) \\ (1/\epsilon)B_2(t) \end{bmatrix}$  

$D(t) = \begin{bmatrix} D_1(t) \\ D_2(t) \end{bmatrix},$  

$D_\epsilon(t) = \begin{bmatrix} D_1(t) \\ (1/\epsilon)D_2(t) \end{bmatrix}$

(36)

And define,

$S(t; \gamma) := B(t)B^T(t) - \frac{1}{\gamma^2} D(t)D^T(t)$

(37)

The GRDE (Generalized Riccati Difference Equation) associated with this game is:

$\tilde{Z} + A_\epsilon^T\tilde{Z} + \tilde{Z}A_\epsilon - \tilde{Z}S_\epsilon\tilde{Z} + Q = 0; \quad \tilde{Z}(t_f) = Q_f$  

(38)

There exists $\gamma^\ast(\epsilon)$ such that for every $\gamma > \gamma^\ast(\epsilon)$, there exists a saddle point controller given by the feedback law,

$u^\ast(t) = \mu^\ast(t, x(t)) = -B_\epsilon^T\tilde{Z}(t; \epsilon)x(t), \quad \gamma \geq 0,$

(39)

**Slow subsystem:**

To obtain the slow dynamics associated with (30), we let $\epsilon=0$, and solve for $x_2$ (to be denoted by $\tilde{x}_2$) in terms of $x_1 = x_{s1}, u = u_s, w = w_s$, and under the working Assumption A3:

$\tilde{x}_2 = -A_{22}^{-1}(A_{21}x_1 + B_2u_s + D_2w_s)$

(40)

Using this fact, the reduced-order (slow) dynamics:

$\dot{x}_s = A_0x_s + B_0u_s + D_0w_s$

(41)

And, the reduced-order (slow) cost is:

$L_u(u, w) = \int_0^{T_f} \left[ x_s(t)_{Q_{11}}^2 + \left( \int_0^T \left[ x_s(t)_{Q_{11}}^2 + x_s(t)_{Q_{22}}^2 + \tilde{x}_2(t)_{Q_{22}}^2 \right] dt \right) + \left[ u_s(t)_{Q_{22}}^2 + u_s(t)_{w_s}^2 \right] dt \right] dt$

(42)

Note that the running cost is not in the standard form because of the cross-terms. However, with appropriate transformations of $u_s, w_s$,

$\dot{x}_s = \tilde{A}_0x_s + \tilde{B}_0(t)\tilde{u}_s + \tilde{D}_0(t)\tilde{w}_s; \quad x_s(0) = 0,$

And the modified cost:

$L_{\tilde{x}} = \int_0^{T_f} \left[ x_s(t_f)_{Q_{11}}^2 + \left( \int_0^T \left[ x_s(t)_{Q_{11}}^2 + \tilde{u}_s(t)_{Q_{22}}^2 + \tilde{w}_s(t)_{Q_{22}}^2 \right] dt \right) dt \right] dt$

(43)

Let, $\tilde{\gamma}_s := \inf \{ \gamma^\ast(\epsilon) \mid \epsilon \geq \tilde{\gamma}_s \}$. It turns out that there exists a saddle point solution to the slow game for $\gamma \geq \tilde{\gamma}_s$ under a more restrictive condition:

$-\frac{1}{\gamma^2} D_2D_2^T + A_{22}Q_{22}^{-1}A_{22}^T > 0$

(44)
Fast subsystem:
In a similar way, one can study fast sub-system. For fast subsystem, let us define,
\[ x_f := x_2 - \bar{x}_2, u_f := u - u_s, w_f := w - w_s, \quad \tau = t' - t / \varepsilon \] (45)
where, \( t \) is frozen and \( t' \) is chosen to vary on the same scales as \( t \).

The reduced –order fast dynamics is given by:
\[ \frac{d}{d\tau} x_f = A_{22} x_f + B_2 u_f + D_2 w_f \] (46)
with the associated cost function:
\[ L^f = \int_0^\infty \left( |x_f'|^2 + |u_f'|^2 + \gamma^2 |w_f'|^2 \right) dt \] (47)

Since, there are no cross-terms, it is easy to solve for the associated GRDE. The feedback controller is given by:
\[ u_f^* (\tau) = \mu_f^* (x_f (\tau)) = -B_2^* (\tau) Z_{ff} (\tau) x_f (\tau) \]
\[ \Rightarrow \quad \mu_f^* (t, x(t)), \quad \gamma_f := \sup_{t \in [0, T_f]} \gamma_f \] (48)

Composite Controller:
The paper now defines a composite controller,
\[ \mu^* (t, x) = \mu^*_y (t, x) + \mu^*_f (t, x) \] (49)
For a composite controller, \( \gamma := \max \{ \gamma_f, \gamma_s \} \). Hence, the associated GRDE attains a non-negative definite solution for \( \gamma \geq \gamma \).

**H∞ Optimal Control in Atomic Force Microscopy**
*(Example problem)*

Feedback Control Design:
Consider the following 1-DOF control system:

![Fig. 3: 1-DOF system with disturbance \( d \) and noise \( n \)](image)

where, \( L = GK \): loop transfer function
\( S = (I+GK)^{-1} = (I+L)^{-1} \): sensitivity function
\( S = (I+GK)^{-1} G K = (I+L)^{-1} L \): complementary sensitivity function

Note that there is an **algebraic limitation**: \( S + T = I \) (51)

Feedback Limitations:
For a perfect control, we want:
\[ e \approx 0. d + 0. r + 0. n \] (52)
This implies that both \( S \) and \( T \) should be zero, contradicting (51). However, the good thing is that the different objectives can be achieved at different frequencies. To shape the Sensitivity and Complementary Sensitivity transfer functions, we use a stacked sensitivity framework,
The closed loop system from augmented disturbance to outputs is given by the equation:

\[
\begin{bmatrix}
z_1 \\
\varepsilon \\
z_3 \\
\end{bmatrix} =
\begin{bmatrix}
-W_s \tilde{G} S & W_s S \\
W_t \tilde{G} S & W_t T \\
-W_t T & W_u K_s S \\
\end{bmatrix}
\begin{bmatrix}
\tilde{d} \\
\varepsilon \\
\tilde{r} - n \\
\end{bmatrix} 
\]

And the objective is: \( \min \| \phi \|_\infty \) \hspace{1cm} (54)

Again, due to fundamental limitation, we want to re-write this associated game as a new-modified game given by the following objective function:

\[
\min_{K, \phi} \left\| W_s \tilde{G} S \right\|_\infty + \frac{1}{m} \left\| \left[ \begin{array}{ccc}
W_s S & W_t T & W_u K_s S \\
\end{array} \right] \phi \right\|_\infty \hspace{1cm} (55)
\]

Scanning probe microscopes refer to the class of devices that use micro-cantilever probes to sense forces and subsequently the properties of samples. The atomic force microscope is a front runner in this family of devices. The probe or micro-cantilever of an AFM gives measurable deflections while sensing forces of \((10^{-7} - 10^{-12})\) N, which has made the measurement of atomic-scale Van-der Waals forces, electrostatic forces, capillary forces and friction forces possible. This ability of force sensing enables nanoscale investigation and manipulation of samples.

For this problem, we assume a cantilever with natural frequency \( \omega_n = 400 \text{ Hz} \), damping factor \( \zeta = 0.008 \). The governing equation for the cantilever is given by:

\[
\frac{d^2 \bar{p}}{dt^2} + 2 \zeta \omega_n \frac{d \bar{p}}{dt} + \omega_n^2 \bar{p} = \omega_n^2 (b \cos(\omega_n t) + u_d) + \frac{1}{m} F(\bar{p} - \bar{h}) + \eta \hspace{1cm} (56)
\]

We treat the van-der Waals force term, \( F(\bar{p} - \bar{h}) \) as the disturbance to this system and the goal is to design an optimal controller that performs this disturbance rejection, simultaneously keeping the reference set-point to \( r = 0 \). The associated game has an optimal \( H^\infty \) norm given by:

\[
\left\| W_s \tilde{G} S \right\|_\infty = 2.15 \hspace{1cm} \text{and} \hspace{1cm} \left\| \begin{bmatrix} W_s S & W_t T & W_u K_s S \end{bmatrix} \phi \right\|_\infty = 3.8 \hspace{1cm} (57)
\]

and the corresponding saddle point feedback control is:

\[
K_i = \frac{0.7141 s^3 + 4012 s^2 + 6333 s + 5835}{s^3 + 47.79 s^2 + 656.6 s + 126.8} \hspace{1cm} (58)
\]

It can be seen that even though, the sample profile varies sinusoidally, the controller is able to reject the disturbance in one cycle of cantilever oscillation, as expected, as the controller has been designed to perform under ‘worst-case scenario’.