Nonlinear Optimal Control

• Control Lyapunov Function (CLF) approach

• Receding Horizon Control (RHC) approach

-- Mayank Baranwal
Optimal Control Problem

Consider the nonlinear dynamics,
\[ \dot{x} = f(x) + g(x)u, \quad f(0) = 0 \]
where, \( x \in \mathbb{R}^n \) denotes the state,
\( u \in \mathbb{R}^m \) denotes the control,
and \( f(x) : \mathbb{R}^n \to \mathbb{R}^n \) and \( g(x) : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are cont. differentiable.

(Optimal Control Problem)

\[
\inf_{u(.)} \int_{0}^{\infty} \left( q(x) + u^T u \right) dt
\]
\[ s.t. \quad \dot{x} = f(x) + g(x)u \]
for \( q(x) \) continuously differentiable and positive semi-definite.
Hamilton-Jacobi-Bellman Equations

The optimal control problem can be reduced to the following PDE:
(also known as HJB equation)

\[ V_x^* f - \frac{1}{4} \left( V_x^* g g^T V_x^* T \right) + q(x) = 0 \]

where, \( V_x^* \) is the **value function** or the **minimum cost-to-go function** form the current state \( x(t) \), i.e.,

\[ V^*(x(t)) = \inf_{u(.)} \int_t^\infty \left( q(x(\tau)) + u(\tau)^T u(\tau) \right) d\tau \]

If there exists a positive definite solution to the HJB equation, then the optimal control action is given by:

\[ u^* = -\frac{1}{2} g^T V_x^* \]
Euler-Lagrange Equations

Local, open-loop approach to optimal control:

\[
\inf_{u(.)} \int_0^T \left( q(x(\tau)) + u(\tau)^T u(\tau) \right) d\tau + \varphi(x(T))
\]

s.t. \[
\dot{x} = f(x) + g(x)u
\]

\[
x(0) = x_0
\]

coincides with optimal control problem when terminal weight \( \varphi(.) \) is \( V^* \)

Calculus of Variation approach yields following necessary conditions:

\[
\dot{x} = H_\lambda(x, u^*, \lambda)
\]

\[
\dot{\lambda} = -H_x(x, u^*, \lambda)
\]

\[
u^* = \arg \min_u H(x,u, \lambda)
\]

where, \( H(x,u, \lambda) = q(x) + u^T u + \lambda^T \left(f(x) + g(x)u \right) \) is Hamiltonian.

\[
x(0) = x_0 \quad (IC)
\]

\[
\lambda(T) = \varphi_x^T (x(T)) \quad (FC)
\]
## Pros and Cons of HJB/E-L approach

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<tr>
<th>HJB Approach</th>
<th>E-L Approach</th>
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<td>Local Approach</td>
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- Pros
  - Global approach
  - Closed-loop solution
  - Solves a restricted problem

- Cons
  - Extremely difficult to solve
Non-exact solutions
(A CLF Based approach)

Let $V$ be the CLF,

$$\dot{V}(x) = V_x(f(x) + g(x)u)$$

Slight variant of **Sontag’s formula**:

$$u_{\sigma_s} = \begin{cases} 
\frac{-V_x f + \sqrt{(V_x f)^2 + q(x)\left(V_x g g^T V_x^T\right)}}{V_x g g^T V_x^T} g^T V_x T & V_x g \neq 0 \\
0 & V_x g = 0
\end{cases}$$

**Q:** Why is Sontag’s formula so **sacrosanct**?

**A:** If CLF has the same level curves as the value function, Sontag’s formula provides the optimal solution.
Proof:- If CLF $V_x$ and value-function $V^*_x$ have the same level-curves,

$$V^*_x = \lambda(x)V_x$$

Substituting value-function into the HJB equation,

$$\lambda V_x f - \frac{1}{4} \lambda^2 (V_x g g^T V_x) + q = 0$$

Solving for $\lambda$, we get,

$$\lambda = 2 \frac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x)}}{(V_x g g^T V_x)}, \quad V_x g \neq 0$$

Solving for optimal solution, $u^*$

$$u^* = -\frac{1}{2} g^T V_x^*$$

$$u^* = -\frac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x)}}{(V_x g g^T V_x)} g^T V_x^T, \quad V_x g \neq 0$$
Pointwise min-norm controllers

\[
\min_{u} u^T u \\
\text{s.t.} \quad V_x (f + gu) \leq -\sigma(x)
\]

Minimize control energy requiring $V$ to decrease at a rate $\sigma(x)$ at every point

For Sontag’s formula,

\[
\sigma_s = \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x^T)}
\]

Note: The resulting controller can be solved for off-line and in closed form
Non-exact solutions
(A RHC Based approach)

\[
\inf_{u(\cdot)} \int_t^{t+T} \left( q(x(\tau)) + u(\tau)^T u(\tau) \right) d\tau + \varphi(x(t + T))
\]

s.t.
\[
\dot{x} = f(x) + g(x)u
\]
\[
x(0) = x_0
\]

- Exploits simplicity of the Euler-Lagrange approach
- On-line implementation is computationally demanding
- Stability issues
Example

Consider the following 2D nonlinear example:

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = -x_1 \left( \frac{\pi}{2} + \arctan(5x_1) \right) - \frac{5x_1^2}{2(1 + 25x_1^2)} + 4x_2 + 3u \]

with performance index:

\[ \int_0^\infty \left( x_2^2 + u^2 \right) dt \]

The optimal solution is known by *converse HJB* method:

\[ V^* = x_1^2 \left( \frac{\pi}{2} + \arctan(5x_1) \right) + x_2^2 \]

And the *optimal control action* is given by:

\[ u^* = -3x_2 \]
Candidate CLF (Control Lyapunov Function)

\[ V = \frac{\pi}{2} x_1^2 + x_2^2 \]
Results

CLF vs HJB approach
Pointwise min-norm controller
(Relation to Sontag’s formula)
Receding Horizon Control
(Performance at different horizon lengths)

At $T = 0.2$

At $T = 0.3$
At $T = 0.4$
At $T = 1.0$
RHC with CLF

Since, the RHC approach suffers from *stability* issues, the two approaches can be combined together to render the system stable

\[
\inf_{u(\cdot)} \int_t^{t+T} \left( q(x) + u^T u \right) d\tau
\]

s.t. \[ \dot{x} = f(x) + g(x)u \]

\[
\frac{\partial V}{\partial x} \left[ f + gu(t) \right] \leq -\epsilon \sigma(x(t))
\]

\[
V(x(t + T)) \leq V(x_{\sigma}(t + T))
\]
At $T = 0.2$

At $T = 1.0$
## Comparisons of CLF/RHC approach

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<td>E-L type of approach</td>
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<tr>
<td>Stability oriented</td>
<td>Performance oriented</td>
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Questions??