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Stabilized interface methods for mechanical joints: Physics-based models and variationally consistent embedding

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This paper presents the application of a new method for interfacial modeling utilizing a merger of continuous Galerkin and discontinuous Galerkin concepts to simulate the behavior of mechanical joints. The interfacial flux terms arising naturally from the discontinuous Galerkin treatment provide a mechanism to embed friction models in a variationally consistent fashion. Due to the unbiased implementation of the interface, facilitated by avoiding the master–slave concept, the deformation of the two interacting surfaces conforms to the local material and geometric attributes of the surfaces. This results in a better preservation of physics in interface mechanics. Additionally, the method is incorporated into a Variational Multiscale framework that comes equipped with a built-in error estimation module, providing numerical estimation of convergence and distinguishing discretization errors from modeling errors. A series of quasi-static numerical simulations of a lap joint under fretting conditions are conducted to compare the performance of two friction models: (i) classical Coulomb friction model and (ii) physics-based multiscale model. Hysteresis study of a three-dimensional double-bolted lap joint for the two friction models is also presented and the computed results are shown to be consistent between conforming and nonconforming meshes.

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1. Introduction

Mechanical joints are an integral component of modern structures and contribute substantially to their serviceability and longevity. Joints transfer forces and moments between connected substructures while introducing localized compliance and dissipation into the overall system. Although ubiquitous in engineering applications, certain aspects of mechanical joint behavior are still not well understood. Examples include the nature of the slipping processes occurring along contacting structural interfaces, the local impacts of mating surfaces remote from the connecting elements (i.e., bolts), and the sensitivity of system response to surface roughness, lubricants, and contaminants. These difficulties increase the uncertainty involved in predicting structural response, resulting in considerable variability even for structures containing nominally identical geometric and material properties.

A fundamental issue hampering the effective simulation of structures containing joints is the prevalence of multiple length scales in the problem. While structural modes of vibration often span lengths on the order of meters, accurate computations often require discretization lengths on the order of centimeters. Additionally, zones of contact may span areas of a few square millimeters, and slip zones can occur along only a fraction of the interface. Finally, surface asperities are of the order of micrometers and the actual mechanisms producing friction occur at atomistic scales, on the order of nanometers. Directly resolving these disparate scales in the context of structural analyses would require mesh densities and minute step time steps that challenge the capacity of current hardware. Thus, the prevalent idea of relying on “larger meshes” or “meshes with increased density” to resolve complex interfacial phenomena has met with limited success in the computational modeling of jointed structures. While the desired outcome from computational technology is predictive science and engineering, current platforms focusing upon “gluing” together various computational facets are failing to live up to expectations. As examples of computational approaches incorporating models for friction that were proposed over the years and have met with varied levels of success, we cite (Oden and Pires, 1984; Wriggers et al., 1990; Ladevéze et al., 2002; Bandeira et al., 2004; Haslinger and Vlach, 2006).

Classical procedures for modeling joints are dominated by traditional computational contact mechanics approaches employing
node-to-surface implementation of contact constraints. The limitations of such approaches for modeling contact between deformable bodies are well known; see for example discussions in Padmanabhan and Laursen (2001), Pei et al. (2005). Since contact tractions are invariably computed via post-processing from nodal forces, such node-to-surface treatments are often incapable of accurately transferring stresses across contact interfaces. This deficiency can lead to pressure oscillations at the interface, which manifests the instability present in the discrete problem and can result in reduced order accuracy for the contact and frictional simulations. Much of the current research for improving contact algorithms has centered on the so-called mortar methods, for which we cite McDevitt and Laursen (2000), Puso and Laursen (2004).

In this work, we propose a variational formulation that emanates from a merger of continuous Galerkin (CG) and discontinuous Galerkin (DG) methods, where the interface is treated as a strong discontinuity similar to the work in Masud et al. (2012). By employing discontinuous functions across the interface, the physical fields are permitted to vary continuously or discontinuously across the jointed surface, which automatically allows for relative slip as dictated by the physics of the system. Continuity of the fields across the interface is weakly enforced via flux terms that arise naturally from the integration by parts of the governing continuum equations. A significant and novel feature of the proposed method is that DG relaxation of the continuity requirement provides a variationally consistent mechanism to incorporate friction models via embedding into these interface flux terms. In this particular work, we incorporate the model proposed in Erten et al. (2010, 2011a) as a constitutive relation for friction at the jointed interfaces. Secondarily, the CG portion of the formulation is enhanced through the Variational Multiscale (VMS) method (Hughes, 1995), which enables the method to seamlessly accommodate both compressible and incompressible materials as well as a wide range of element types. The method is also equipped with a built-in error estimator that provides instant feedback on the numerical accuracy of computed solutions. The combination of these methods produces a robust framework for simulating structural systems containing mechanical joints. This framework is investigated within the context of numerical tests for two and three-dimensional models of bolted lap joints, and the computed results are seen to be consistent between conforming and nonconforming meshes.

In the remainder of the paper, we first summarize the stabilized interface formulation in Section 2 and discuss its relation to traditional contact algorithms. Section 3 provides an overview of the physics-based friction model. The incorporation of the model into the interface formulation and the associated nonlinear solution procedure is described in Section 4. In Section 5, the performance of the friction model is compared to results obtained using classical Coulomb friction within the formulation for two and three-dimensional numerical tests. Concluding remarks concerning the method are given in Section 6.

2. Underlying variational framework

2.1. Interfacial coupling and stabilization strategies

We primarily focus on fretting contact for which the location of the contact interface is known a priori. Accordingly, we consider the problem from the perspective of a single body with a predefined interface. This exposition summarizes the relevant developments of Masud et al. (2012). Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded domain with a piece-wise smooth boundary \( \Gamma \), where \( n_{ad} \geq 2 \) is the number of spatial dimensions. The boundary \( \Gamma \) is divided into two subsets \( \Gamma_g \) and \( \Gamma_h \) on which Dirichlet and Neumann conditions are applied, respectively, and these subsets satisfy \( \Gamma_g \cup \Gamma_h = \Gamma \), \( \Gamma_g \cap \Gamma_h = \emptyset \). With these definitions, the governing equations of linear elasticity are:

\[
\nabla \cdot \sigma + b = 0 \quad \text{in} \quad \Omega \tag{1}
\]

\[
u = g \quad \text{on} \quad \Gamma_g \tag{2}
\]

\[
a \sigma = h \quad \text{on} \quad \Gamma_h \tag{3}
\]

where \( \mathbf{u} : \Omega \rightarrow \mathbb{R}^{3 \times 3} \) represents the displacement field, \( \sigma \) is the Cauchy stress tensor, \( b \) is the body force vector, \( g \) is the prescribed displacement, \( h \) is the prescribed traction, and \( \mathbf{n} \) is the unit outward normal to \( \Gamma \). All through the paper, the following conventions are used for vector and tensor operators: \( \nabla (\cdot) \) represents the gradient, \( \nabla \cdot (\cdot) \) is the divergence, and \( (\cdot)^\top \) is the transpose of the indicated quantity.

Multiplying by the weighting function \( \mathbf{w} \) and integrating over the domain, we arrive at the weighted-residual form, where internal stresses balance the externally applied loads:

\[
\int_\Omega \mathbf{w} : \nabla \cdot \sigma + \mathbf{w} : b \, d\Omega + \int_{\Gamma_h} \mathbf{w} \cdot (h - a \mathbf{n}) \, d\Gamma = 0 \tag{4}
\]

This form is then applied to a discretization of the domain \( \Omega \) consisting of disjoint open subdomains indicated by \( \Omega^e \) and element boundaries by \( \Gamma^e = \{1, \ldots, n_{\text{elem}}\} \), arranged such that \( \Omega = \bigcup_{e=1}^{n_{\text{elem}}} \Omega^e \). Applying integration by parts followed by the divergence theorem to the stress term gives rise to a sum of terms on element interiors and element boundaries:

\[
\sum_{e} \int_{\Omega^e} \left[ \nabla \mathbf{w}^e : \sigma + \mathbf{w}^e : b \right] \, d\Omega + \sum_{e} \int_{\Gamma^e} \mathbf{w}^e : \mathbf{n} \, d\Gamma = 0 \tag{5}
\]

Along the inter-element boundaries on the interior of the domain, the second integral term involves contributions from elements sharing a common surface. For a conforming mesh, these element boundary integrals cancel pairwise because the continuous discrete variational displacement field \( \mathbf{w}^e \) enforces traction equilibrium between elements in a weak sense.

Now, we introduce an interface separating the domain into two regions, designated + and - as in Fig. 1. If the element edges on both sides of the interface do not match up, the mesh is nonconforming, and displacement continuity along with traction equilibrium are no longer guaranteed. Thus, the boundary integrals in (5) do not necessarily vanish and give rise to un-equilibrated flux terms:

![Fig. 1. Single domain with imposed interface.](Image)
\[
\sum_{e} \int_{V_e} [-\nabla \mathbf{w} : \mathbf{\sigma} + \mathbf{w} : \mathbf{b}] \mathbf{d} \Omega + \sum_{e} \int_{\Gamma_\text{int}} \mathbf{w} : \mathbf{\sigma} \mathbf{n} \mathbf{d} \Gamma + \sum_{e} \int_{\Gamma_\text{int}} \mathbf{w} : \mathbf{\sigma} \mathbf{n} \mathbf{d} \Gamma = 0
\] (6)

Consequently, two constraints need to be enforced at the interface: (i) continuity of displacements, and (ii) equilibrium of tractions. To satisfy these constraints, we incorporate interface stabilization terms emanating from the discontinuous Galerkin method as proposed in Masud et al. (2012). The key idea is to introduce two numerical flux terms that weight the violation of these conditions. The first term is the average weighting function times the jump in traction, and the second is the average variational flux times the jump in displacement. Adding these two terms into the weak form (6) produces a symmetric form enforcing continuity and equilibrium along the interface systematically without the introduction of auxiliary Lagrange multipliers:

\[
\sum_{e} \int_{V_e} \nabla \mathbf{u} : \mathbf{\sigma} \mathbf{d} \Omega - \int_{V_e} [\mathbf{w}] : \{\sigma \mathbf{u}\} \mathbf{d} \Gamma - \int_{\Gamma_\text{int}} \{\sigma \mathbf{w}\} : [\mathbf{u}] \mathbf{d} \Gamma = 0
\]

where the jump and average operators are defined along the interface for a scalar field \(a\), vector \(\mathbf{b}\), or tensor \(\mathbf{C}\) as follows:

\[
[a] = a^+ - a^-, \quad [\mathbf{C}] = \mathbf{C}^+ - \mathbf{C}^-
\]

\[
[b] = b^+ \mathbf{n} + b^- \mathbf{n}, \quad [\mathbf{b}] = \mathbf{b}^+ \mathbf{n} + \mathbf{b}^- \mathbf{n}
\]

\[
\{a\} = (a^+ + a^-)/2, \quad \{\mathbf{b}\} = (\mathbf{b}^+ + \mathbf{b}^-)/2, \quad \{\mathbf{C}\} = (\mathbf{C}^+ + \mathbf{C}^-)/2
\]

Remark. In order to ensure numerical stability, a third term of the form \(\sum_{e} \int_{\Gamma_\text{int}} \varepsilon [\mathbf{w}] : [\mathbf{u}] \mathbf{d} \Gamma\) is required, which ensures the positive-definiteness of the discrete stiffness matrix. The exact form of this term is presented below.

Next, we elevate this pure-displacement interface formulation to a mixed form with enhanced stability accommodating both compressible and incompressible materials through the incorporation of the Variational Multiscale method (Hughes, 1995), as presented in Masud and Xia (2005, 2006), Masud et al. (2011). The key idea in this approach is to postulate a unique additive decomposition of the solution fields into coarse and fine scales, where the coarse-scale field represents the standard finite element solution while the fine-scale field models features that are beyond the resolution capacity of a given mesh or discretization:

\[
\mathbf{u} = \mathbf{u}_\text{coarse} + \mathbf{u}_\text{fine scale}
\]

\[
\mathbf{w} = \mathbf{w}_\text{coarse} + \mathbf{w}_\text{fine scale}
\]

Accounting for these fine-scale features in the computed coarse scales enhances the stability of the variational form, which ultimately allows for the use of equal-order polynomial interpolations for the displacement and pressure fields. The mixed constitutive law for linear elasticity along with the compatibility and kinematic equations are given as:

\[
\mathbf{\sigma} = \sigma(\mathbf{u}, p) = \mathbf{p} l + 2 \mu \mathbf{e}(\mathbf{u})
\]

\[
\nabla \cdot \mathbf{u} = p/\mathcal{E}
\]

\[
\varepsilon = \varepsilon(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)
\]

where \(p : \mathbf{\Omega} \rightarrow \mathbb{R}\) denotes the pressure field, \(\varepsilon\) is the linear strain tensor, \(\mathcal{E}\) and \(\mu\) are the Lamé parameters, and \(\mathcal{I}\) is the second-order identity tensor. Substituting these relations into the interface weak form (7) produces a mixed formulation. The Variational Multiscale approach begins by incorporating the decomposition of the displacement field (10) and (11) into the mixed form. Next, modeling assumptions are applied whereby the fine scales are approximated by bubble functions \(b^c\), which permits localization of the equations for the fine-scale terms to element interiors and also ensures that the fine scales do not affect the interface stabilization terms defined on element boundaries as presented above. Once a specific choice is made for the bubble functions, an expression can be derived for \(\mathbf{u}\) on each element as a function of the coarse-scale residual; the details of this derivation can be found in Masud et al. (2011). The resulting equation for the fine scales is:

\[
\mathbf{u}^h = \tau (\nabla p + 2\mu \nabla \cdot \varepsilon(\mathbf{u}) + \mathbf{b})
\]

where \(\tau\) is a second-order stabilization tensor with the following form:

\[
\tau = \left( \left( \int_{Q^f} \mu \nabla b^c \cdot \nabla b^c \mathbf{d} \Omega \right) \mathcal{I} + \int_{Q^f} \mu \nabla b^c \otimes \nabla b^c \mathbf{d} \Omega \right)^{-1}
\]

Finally, these expressions for the fine scales can be substituted into the mixed form to account for their effects on the coarse-scale fields while removing their explicit appearance. Combining the domain-based and interface stabilization terms, the final form of the stabilized mixed weak form accommodating embedded interfaces is given by:

\[
\int_{Q^c} 2\mu \mathbf{e}(\mathbf{w}) : \varepsilon(\mathbf{u}) + (\nabla \cdot \mathbf{w}) p + q(\nabla \cdot \mathbf{u}) - q_p \mathbf{d} \Omega
\]

\[
- \int_{Q^c} \nabla q + 2\mu \nabla \cdot \varepsilon(\mathbf{w}) \cdot \mathbf{\tau} (\nabla p + 2\mu \nabla \cdot \varepsilon(\mathbf{u})) \mathbf{d} \Omega
\]

\[
- \int_{Q^f} \{\mathbf{p} + 2\mu \mathbf{e}(\mathbf{u})\} + \{q\mathbf{I} + 2\mu \mathbf{e}(\mathbf{w})\} : [\mathbf{u}] \mathbf{d} \Gamma
\]

\[
+ \int_{Q^f} \varepsilon(\mathbf{\mu}) \mathbf{h} : [\mathbf{w}] : [\mathbf{u}] \mathbf{d} \Gamma = \int_{Q^c} \mathbf{w} : \mathbf{b} \mathbf{d} \Omega + \int_{Q^f} \mathbf{w} \cdot \mathbf{h} \mathbf{d} \Gamma
\]

\[
+ \int_{Q^c} \nabla q + 2\mu \nabla \cdot \varepsilon(\mathbf{w}) \cdot \mathbf{\tau} \mathbf{d} \Omega
\]

where the domain integrals are evaluated on the union of element interiors \(\Omega = \bigcup_{e}^{\text{fine}} \Omega_e\). The last term on the left-hand side is the penalty term ensuring numerical stability of the formulation, where the definition of the element length scale \(\mathcal{h}\) is defined based on the size of the local contacting elements, given for any pair of elements as:

\[
h_j = 2 \left( \frac{\text{meas}(\gamma^+_j)}{\text{meas}(\gamma^-_j)} + \frac{\text{meas}(\gamma^-_j)}{\text{meas}(\gamma^+_j)} \right)^{-1}
\]

where \(\gamma^+_j\) and \(\gamma^-_j\) are elements from regions \(\Omega^+\) and \(\Omega^-\) bordering \(\Gamma_j\), and \(\gamma^+_j\) and \(\gamma^-_j\) are the entire element faces lying on the interface. Specifying a value of the penalty parameter \(\varepsilon \in [5, 100]\) has been found to provide stable and accurate numerical results for all combinations of element types (Masud et al., 2012).

Remark. For a nonconforming mesh in the discrete setting, the proper evaluation of the interface term requires the partitioning of the interface into integration segments defined over shared portions of element edges. Along each segment, numerical integration is performed to evaluate contributions from two particular elements on the \(\Gamma^+\) and \(\Gamma^-\) faces. For implementational details see (Masud et al., 2012).
Remark. The formulations given in (7) and (17) are capable of representing linear models for interface behavior through substitution in place of the Cauchy stress tensor in the boundary integrals. For inclusion of nonlinear interface behavior, the reader is referred to Section 4. The discussion there is applicable both to a pure displacement-based method of (7) or the mixed field treatment of (17).

2.2. Extension to contact problems

Now we want to illustrate the relationship of our formulation (17) to traditional contact algorithms. The governing equations from two body contact problems involve constraint equations for surface normal interactions, called the Kuhn-Tucker conditions, that describe the relationship between the penetration or gap $g_N$ and the contact pressure $t_N$ (Wriggers, 2006). Coupling in the tangential direction is expressed through conjugate fields termed the tangential gap $g_T$ and the shearing traction $t_T$. The gap functions are defined as follows:

$$g_N = -[X + u] \cdot n' - g_N n'$$

The Kuhn–Tucker contact conditions are incorporated into the standard weak form via the integral:

$$\int_{\Omega} \nabla w : \sigma d\Omega + \int_{\Gamma_1} \delta g_N \cdot t_N + \delta g_T \cdot t_T d\Gamma = \int_{\Omega} w \cdot b d\Omega + \int_{\Gamma_1} w \cdot h d\Gamma$$

where $\delta g_N$ and $\delta g_T$ are the variational gaps in the normal and tangential directions, respectively (Wriggers, 2006). Observing the form of the first interface stabilization term in (17), we note the similarity with the traditional contact terms. In particular, both expressions involve the multiplication of the variational gap $w$ by some measure of the interface traction. This suggests using the discontinuous Galerkin terms to enforce the contact conditions of (22).

Substituting the component relation (21) into the interface formulation (17), we obtain the final weak form for contact problems involving friction:

$$\int_{\Omega} 2 \mu \epsilon(w) : \epsilon(u) + |\nabla \cdot w|^p + q |\nabla \cdot u| - q p / q d\Omega$$

Remark. The combination of the stabilization terms and the incorporation of the interface constitutive equations directly into an integral expression provide a variationally consistent method for modeling interfacial phenomena.

2.3. Embedded error estimation

Before proceeding to the friction model, we reconsider the multiscale decomposition of the displacement field: $u = u + u'$. Ranging for $u'$, we observe that the fine-scale field represents the relative error between the exact solution and the coarse-scale field: $u = u - u'$. Therefore, a second role for the fine scales emerges: $u'$ can serve as an indicator for how well the finite element solution, $u' \approx u$, approximates the exact solution to the governing equations, $u$. This error is commonly referred to as the discretization error:

$$e = u - u'$$

Thus, a good approximation of $u'$ can serve as a reasonable indicator of the discretization error. The expression for $u'$ using bubble functions given by (15) represents one such approximation. Once the finite element solution has been obtained by solving (22), that expression can be explicitly evaluated to obtain a measure of the error:

$$e = \tau |\nabla p^h + 2\mu |\nabla \cdot (\epsilon(u'))| + b$$

The quality of the solution can then be evaluated by taking a norm of the fine-scale field:

$$||e||_{L^2(\omega)} = \sqrt{\int_{\omega} \epsilon \cdot \epsilon d\Omega}$$

where $\omega$ can be either an element $\Omega^e$ for local evaluation or $\Omega^g$ for global evaluation. The performance of this error indicator has been assessed for single domain problems in (Masud et al., 2011) and nonconforming meshes and contact problems in (Masud et al., 2012).

Remark. Error estimation is important both for quantifying the accuracy of a numerical solution and for isolating discretization error from modeling error. Modeling error reflects the inaccuracies in material parameters or boundary conditions as well as the discrepancy between the governing equations and the physics of the actual system. In particular, once the discretization error is made arbitrarily small, any residual discrepancies between the numerical solution and experimental results can be attributed to the modeling error. Incorporating such error estimators allows finite element simulations to serve as a tool to assist in the validation of new models, such as the one subsequently proposed.

3. Contact modeling

Nominally flat contacts are found in many applications such as shrink fits, bolted joints, turbine blades and brakes. Modeling fretting of a smooth flat-on-flat contact is a mathematically hard problem due to singularities at the contact edges (Hills and Nowell, 1994). In addition, it is nearly impossible to machine a perfectly (or infinitely) smooth engineering surface (except for mica surfaces). Surface geometry involves micro irregularities, calledasper-
As mentioned above, when two nominally flat rough surfaces are brought into contact, the contact occurs at the asperity tips, and a gap $d$ forms between the reference planes defined by the mean of the asperities and the rigid flat. Hence, only the asperities with heights greater than that gap (i.e., its interference is positive, $z_i + z - d > 0$) carry the entire load. If one expresses the height distribution of the asperities by a probability density function, $\phi(z)$, the percentage of the asperities making contact and the total load they carry can be found statistically. This approach – GW Model – was first introduced in (Greenwood and Williamson, 1966) as the statistical summation approach. In this approach, the rough surface is assumed to be isotropic over a nominally flat area $A_0$, which consists of spherical asperities with uniform areal density $\eta$ and radius $R$. In addition, asperities are assumed to be distributed far apart over the contact surface, with no interaction between them. In the presence of these assumptions, individual asperity-scale contacts can be modeled after Hertzian contact (Hertz, 1882) as follows:

$$P_{asp}(z, d) = \frac{4}{3} \sqrt{RE} \sigma^{3/2}(z - d)^{3/2}$$

where $E = \left[\left(1 - v_1^2\right)/E_1 + \left(1 - v_2^2\right)/E_2\right]^{-1}$ is the combined Young’s modulus, $\sigma$ is the standard deviation of asperity height distribution, $z$ is the height of asperities normalized with respect to $\sigma$, and $d$ is the normalized gap as described above.

Among various asperity height distributions observed on engineering surfaces, the normal distribution is the most common for surfaces manufactured by abrasive and/or generic cumulative removal processes. Although running-in surfaces (produced by honing, lapping and superfinishing) usually exhibit asymmetrical surface height distributions (such as skewed and bimodal), asperity heights continue to exhibit distributions quite close to normal (Kikuchi and Oden, 1988; Dowson, 1989; Stolarski, 2000; Yu and Polycarpou, 2002). Noting that simple distributions yield straightforward analytical expressions, we can approximate the normal distribution as a triangular distribution with identical standard deviation as follows:

$$\phi(z) = \frac{1}{\sqrt{6}} \begin{cases} 1 + \frac{z}{\sqrt{6}} & -\sqrt{6} \leq z < 0 \\ 1 - \frac{z}{\sqrt{6}} & 0 \leq z \leq \sqrt{6} \end{cases}$$

Assuming that the gap between the mean of asperity heights and the rigid flat remains positive, i.e., $d > 0$, the statistical summation procedure yields the following expression for normal contact load:

$$P_{tot}(d) = \eta A_0 \int_{-d}^{d} P_{asp}(z, d) \phi(z) dz = \frac{8}{315} \eta A_0 \sqrt{RE} \sigma^{3/2}(\sqrt{6} - d)^{3/2}$$

3.1. Normal contact of nominally flat rough surfaces

Fretting contact occurs when two bodies in contact are preloaded by a normal load and slide against each other in the tangential direction. Constant normal preload followed by cyclic tangential loading is a simple subset of fretting contact loading conditions. Evaluating a combined roughness for the contacting rough surfaces and representing the contact as rigid flat-on-rough contact further simplifies the theoretical treatment of the problem, as depicted in Fig. 2.

![Fig. 2. Schematic representation of an equivalent rough surface and a rigid flat contact and, probability density function for the asperity heights.](image)
3.2. Tangential contact of nominally flat rough surfaces

The response of a flat-on-flat contact to tangential loading can have two different regimes: (i) partial slip, when part of the contact interface slides and the remaining part sticks; (ii) gross sliding, when the entire contact slides. In the latter case, all of the asperities exhibit gross sliding behavior, and the total tangential force can be calculated with the statistical summation approach as in normal loading:

\[ Q_{\text{flat}}^{\max}(d) = \eta A_0 \mu \int_{-d}^{d} P_{\text{asp}}(z, d)\phi(z) \, dz \]  

Note that if a local Coulomb friction law holds at the asperity-scale, the coefficient of friction in the integrand of (29) is constant, and the contact model at the macroscale becomes identical to the Coulomb friction model:

\[ Q_{\text{flat}}^{\max}(d) = \mu \eta A_0 \int_{-d}^{d} P_{\text{asp}}(z, d)\phi(z) \, dz = \mu P_{\text{flat}}(d) \]  

When the tangential load is insufficient to cause gross sliding, the response of each asperity is coupled to the normal loading. Shorter asperities carry less normal load and, hence, slide more easily than taller asperities. Also, the loading history affects the tangential response of each asperity; i.e., individual asperity responses to loading, unloading and reloading differ due to history-dependence of tangential loading. Fig. 3 shows the probability density function of asperity heights divided into regions of different tangential responses. The rigid flat is pushed until its distance to the mean of heights is \( d \), and thus only asperities with heights greater than \( d \) are in contact.

As shown in Fig. 3, contacting asperities can be separated into two groups while loading and three groups while unloading according to the asperities’ response to loading—unloading as described in (Björklund, 1997; Ertien et al., 2011b). Under initial loading, while taller asperities stick and contribute to the tangential stiffness of the contact, shorter asperities slip and cause frictional energy dissipation. A subsequent unloading of the contact results in three different asperity behaviors. The asperities which stick through initial loading remain stuck while unloading, since the contact force on these asperities prevents any slippage. In contrast, while unloading, the asperities which slip while loading can either slip or stick, depending on their heights. This is mainly due to a decrease in tangential force upon unloading. Since the preload on these asperities is assumed to be constant throughout tangential loading/unloading, the limiting tangential force is constant. Therefore, a decrease in tangential force results in some of the asperities carrying tangential load less than the limiting force, and, thus, they start sticking while unloading. As expected, the shorter asperities among those previously slipping asperities continue slipping, whereas the taller asperities stick. It is customary to model sticking asperities as partial slip contact with a gross sliding limit of \( \mu P \). The tangential force in partial slip contact of elastic spheres under loading by tangential displacement \( \delta \) was introduced first in (Cattaneo, 1938) and later independently in (Mindlin, 1949) as follows:

**Partial slip (loading):**

\[ Q_{\text{asp}}(z, d, \delta) = \mu P_{\text{asp}}(z, d) \left\{ \begin{array}{ll}
1 - \left( \frac{4G\delta}{\mu E \sqrt{d}} \right)^{3/2} & z \leq z_0 \\
1 & z > z_0
\end{array} \right. \]  

where \( z_0 = d + 4G\sqrt{\delta/\mu E} \) is the limiting height and \( G = [(2 - v_1)/G_1 + (2 - v_2)/G_2]^{-1} \) is the combined shear modulus. The statistical summation utilizing the triangular distribution of asperity heights gives the tangential force for flat on flat contact as:

\[ Q_{\text{flat}}^{\text{load}}(d, \delta) = \eta A_0 \int_{-d}^{d} Q_{\text{asp}}(z, d, \delta)\phi(z) \, dz \]

\[ = \mu \eta A_0 \left[ 1 - \left( \frac{4G\delta}{\mu E \sqrt{d}} \right)^{3/2} \right] \]  

While unloading, three different responses for the individual asperities can be modeled as:

**Partial slip (unloading):**

\[ Q_{\text{asp}}(z, d, \delta, \delta_{\max}) = \mu P_{\text{asp}}(z, d) \left\{ \begin{array}{ll}
-1 & z \leq z_1 \\
2 \left( 1 - 2G\sqrt{\delta_{\max} - \delta}/\mu E \sqrt{d} \right)^{3/2} - 1 & z_1 < z < z_2 \\
2 \left( 1 - 2G\sqrt{\delta_{\max} - \delta}/\mu E \sqrt{d} \right)^{3/2} - \left( 1 - 4G\delta_{\max}/\mu E \sqrt{d} \right)^{3/2} - 1 & z_2 < z
\end{array} \right. \]  

Here, \( \delta_{\max} \) is the maximum tangential displacement (tangential displacement at maximum tangential force) and limiting heights \( z_1 = d + 2G\sqrt{\delta_{\max}}/\mu E \), \( z_2 = d + 4G\delta_{\max}/\mu E \). Statistical summation gives the tangential force for nominally flat surfaces under tangential unloading as:

\[ Q_{\text{flat}}^{\text{unload}}(d, \delta, \delta_{\max}) = \eta A_0 \int_{-d}^{d} Q_{\text{asp}}(z, d, \delta, \delta_{\max})\phi(z) \, dz \]

\[ = \mu \eta A_0 \int_{-d}^{d} \left[ 2 \left( 1 - 2G\sqrt{\delta_{\max} - \delta}/\mu E \sqrt{d} \right)^{3/2} - \left( 1 - 4G\delta_{\max}/\mu E \sqrt{d} \right)^{3/2} - 1 \right] \]  

Assuming that the same events occur in the reverse direction while unloading, the response can be found simply by reversing the direction of both force and displacement in (34).

**Partial slip (reloading):**

\[ Q_{\text{flat}}^{\text{unload}}(d, \delta, \delta_{\max}) = -Q_{\text{flat}}^{\text{load}}(d, -\delta, \delta_{\max}) \]  

![Fig. 3. Asperity behavior in tangential: (a) loading, (b) unloading.](image-url)
3.3. Discussion of physics-based modeling

The contact model proposed in this work employs Mindlin’s formulation for fretting behavior of sphere-on-flat contact (Mindlin et al., 1952) for the response of each individualasperity and the sum of these responses using the statistical framework proposed by Greenwood and Williamson to obtain the fretting response for elastic/plastic contact of nominally flat rough surfaces (Greenwood and Williamson, 1966). This modeling approach was utilized by other researchers (Björklund, 1997; Berthoud and Baumberger, 1998; Cohen et al., 2008; Eriten et al., 2011b). Note that this approach needs a predetermined constant coefficient of friction at theasperity scale. Eriten et al. (2010) proposed a physics-based friction and elastic–plastic fretting model forasperity level. Having this model in hand, we considered the problem where load is applied incrementally in a series of steps. One load step is defined as the loading of previous load step. In the following sections, we will describe the representations for the interface treatments using discrete springs or node-on-node contact.

Remark. Due to the nonlinearity of the friction models and theinitially undetermined zones of stick and slip, the tangentialinterface term $\partial \tau \cdot \mathbf{f}_t$ causes the system of equations (36) to become nonlinear. In the discrete setting, we employ the Newton–Raphson algorithm to solve this system of equations in an iterative fashion. Thus, to proceed from step $n-1$ to step $n$, the linearizedform of (36) must be solved over a series of iterations $i=1,2,\ldots$, until the unknown quantities converge to within a specified tolerance. These values are then recorded as the converged equilibrium state.

Remark. In the numerical simulations in Section 5, we take the definition of the contact pressure to be $\mathbf{t}_n^{(c)} = \sigma_n^{(c)} + \mathbf{e} \mathbf{f}_t^{(c)} / h_n^{(c)}$ which was found to be a more robust measure for making contact/gap determinations than the average numerical pressure $\sigma_n$ alone.

4. Variational embedding of the interface models

We now discuss the incorporation of two specific interface models into the interface formulation derived in Section 2: (i) direct application of the classical Coulomb law and (ii) the physics-based model presented in Section 3. This formulation provides a variational setting for imposing the physical constraints in an unbiased fashion, as opposed to the classical master–slave algorithms encountered in practice. Herein, the interface models are substituted in place of the Cauchy stress tensor in the tangential interface term of (22), providing a mechanism that enforces the constitutive relationship in an integral sense. This approach results in a variational form with improved consistency compared to treatments using discrete springs or node-on-node contact.

Both interface models under consideration are nonlinear and history-dependent. Therefore, expression (22) must be integrated in time to obtain the response of the domain during the loading history. For slowly-varying loads, inertial effects can be neglected, and a quasi-static numerical implementation can be adopted. Thus, we consider the problem where load is applied incrementally in a series of steps $n=1,2,\ldots,n_{\text{step}}$. A backward Euler integration scheme is applied to solve (22) for the equilibrium state corresponding to each load level. Considering an arbitrary load step $n$ for which the preceding equilibrium state is known, equation (22) takes the following form:

### Remark

\[
\varepsilon_t^N > 0 
\]

where the contact pressure $t_0$ is computed as in Section 2.2.

**Remark.** While the original Coulomb model specifies no relative motion in the stick zone, experiments have shown partial slip for joints in the stick region, indicating that regularized laws may be a closer approximation of physical reality (Wriggers et al., 1990). This particular regularized relationship has an analogy with elastic-plastic material behavior. The parameter $\varepsilon_r$ therefore is a reflection of the tangential stiffness of the interface. However, its value is left unspecified by the theory and must be estimated by fitting numerical solutions to experimental data.

The above equations can be cast into an incremental form by employing a return mapping strategy whereby a trial “stick” state is computed and subsequently projected onto the actual state. Starting from the last converged state $n-1$, the advancement of the interface quantities are given by the following expressions:

### Remark

Starting from the last converged state $n-1$, the advancement of the interface variables can be described as follows.

\[
\mathbf{t}_n^{(c)} = \mathbf{t}_n^{(c)(n-1)} + \varepsilon_r \left( \mathbf{g}_n^{(c)} - \mathbf{g}^{(c)(n-1)}_n \right) 
\]

### Remark

The contact pressure $t_0$ is computed as in Section 2.2.

**Remark.** While the original Coulomb model specifies no relative motion in the stick zone, experiments have shown partial slip for joints in the stick region, indicating that regularized laws may be a closer approximation of physical reality (Wriggers et al., 1990). This particular regularized relationship has an analogy with elastic-plastic material behavior. The parameter $\varepsilon_r$ therefore is a reflection of the tangential stiffness of the interface. However, its value is left unspecified by the theory and must be estimated by fitting numerical solutions to experimental data.

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**Remark.** In the numerical simulations in Section 5, we take the definition of the contact pressure to be $\mathbf{t}_n^{(c)} = \sigma_n^{(c)} + \mathbf{e} \mathbf{f}_t^{(c)} / h_n^{(c)}$ which was found to be a more robust measure for making contact/gap determinations than the average numerical pressure $\sigma_n$ alone.

### 4.1. Classical Coulomb friction

For the case of Coulomb friction, we adopt the penalty regularization presented in Simo and Laursen (1992). The rate form of this model is specified by the following Kuhn–Tucker conditions:

\[
\Phi = ||\mathbf{t}_t|| - \mu_t t_0 < 0
\]

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The above equations can be cast into an incremental form by employing a return mapping strategy whereby a trial “stick” state is computed and subsequently projected onto the actual state. Starting from the last converged state $n-1$, the advancement of the interface quantities are given by the following expressions:

**Remark.** Due to the nonlinearity of the friction models and the initially undetermined zones of stick and slip, the tangential interface term $\partial \tau \cdot \mathbf{f}_t$ causes the system of equations (36) to become nonlinear. In the discrete setting, we employ the Newton–Raphson algorithm to solve this system of equations in an iterative fashion. Thus, to proceed from step $n-1$ to step $n$, the linearized form of (36) must be solved over a series of iterations $i=1,2,\ldots$, until the unknown quantities converge to within a specified tolerance. These values are then recorded as the converged equilibrium state.
From these expressions, we observe that the quantities \( t_{n-1}^{(n)} \) and \( g_{n-1}^{(n)} \) from the previous load step serve as the state variables for this model. The relations for \( t_{n}^{(n)} \) given by (45) and (46) are substituted into (36) to give the weak form including Coulomb friction at the interface. This nonlinear equation is solved iteratively for the increments \( \Delta u \) and \( \Delta p \) that update the equilibrium configuration to load level n.

### 4.2. Physics-based friction model

Compared to the classical Coulomb law, the model presented in Section 3 is more amenable to a load branch tracing strategy. Therefore, we consider the domain to be subjected to a series of loads that cause the interface to experience a sequence of loading and unloading events. Each loading branch is designated by a counting parameter \( s = 1, 2, \ldots \).

**Remark.** For structures containing multiple interfaces experiencing complicated loading, different interfaces may undergo loading and unloading conditions concurrently. Thus, each local region of an interface should be treated as having a separate loading branch sequence.

We begin by focusing on the initial loading branch for \( s = 1 \) which occurs along a portion of the interface immediately following first contact during the subsequent load step. This condition corresponds to the loading Eq. (32) from Section 3.2. Locally, we associate the tangential traction \( t_{n}^{(1)} \) with the quantity \( Q_{n}^{\text{load}}/A_{k} \). However, Eq. (32) requires a value for the mean normal interference \( d \). Therefore, we must first obtain this quantity from the expression for the normal force (28). Solving this equation for \( d \), we substitute the normal contact pressure \( t_{n}^{(1)} \) for \( K_{n}^{\text{load}}/A_{k} \), and obtain the following:

\[
d = \sqrt{6} - \left( \frac{215r_{n}^{(1)}}{8eR^{2}E^{\alpha^{2}}} \right)^{2/7} \tag{47}\]

Then, we can evaluate (32) replacing the tangential displacement \( \delta \) with the magnitude of \( g_{n}^{(1)} \), which results in

\[
t_{n}^{(1)} = \mu_{t}^{(1)} t_{n}^{(1)} \frac{g_{n}^{(1)}}{|g_{n}^{(1)}|} \left[ 1 - \frac{1}{\mu_{t}^{(1)} E} \left| \frac{g_{n}^{(1)}}{|g_{n}^{(1)}|} \right| \sqrt{6 - d} \right]^{7/2} \tag{48}\]

Finally, when the value of \( |g_{n}^{(1)}| \) reaches the critical value \( \mu_{t}^{(1)} E (\sqrt{6} - d)/4G \), the maximum tangential force \( |t_{n}^{(1)}| = \mu_{t}^{(1)} t_{n}^{(1)} \) is reached, and any larger deformations are then evaluated by the slip condition (46) setting \( g_{n}^{(1)} = 0 \). This procedure for evaluating \( t_{n}^{(1)} \) is summarized in Table 1.

For subsequent load steps, either (48),(46) is used so long as the interface continues to be loaded, i.e. \( |g_{n}^{(1)}| \gg |g_{n}^{(1)}| \). Once this inequality is violated, the unloading phase begins; thus, the value of \( s \) is increased, and the extreme value of the gap is stored as \( g_{n}^{(1)} \). The tangential traction is now associated with the quantity \( Q_{n}^{\text{load}}/A_{k} \) with analogy to (33) and (34) from Section 3.2. Thus, the formula for \( t_{n}^{(s)} \) becomes:

\[
t_{n}^{(s)} = K_{n}^{(s)} \mu_{t}^{(s)} t_{n}^{(s)} \frac{g_{n}^{(s)} - g_{n-1}^{(s-1)}}{|g_{n}^{(s)}| - |g_{n-1}^{(s-1)}|} \tag{49}\]

where the factor \( K_{n}^{(s)} \) accounts for the interface stick/slip condition:

\[
K_{n}^{(s)} = \begin{cases} 
1 & \text{if } d < d_{1} \\
1 - \frac{2G}{\mu_{t}^{(n)} E} \sqrt{6 - d} \left| \frac{g_{n}^{(n)}}{|g_{n}^{(n)}|} \right|^{7/2} & \text{if } d_{1} \leq d < d_{2} \\
1 - \frac{2G}{\mu_{t}^{(n)} E} \sqrt{6 - d} \left| \frac{g_{n}^{(n)}}{|g_{n}^{(n)}|} \right|^{7/2} + \left[ 1 - \frac{4G}{\mu_{t}^{(n)} E} \sqrt{6 - d} \right]^{7/2} & \text{if } d_{2} \leq d
\end{cases} \tag{50}\]

and the limiting approach values are \( d_{1} = \sqrt{6} - (2G/\mu_{t}^{(n)} E) |g_{n}^{(n)} - g_{n-1}^{(n-1)}| \) and \( d_{2} = \sqrt{6} - (4G/\mu_{t}^{(n)} E) |g_{n}^{(n)} - g_{n-1}^{(n-1)}| \).

The region of the interface continues along the unloading branch as long as \( |g_{n}^{(n)}| \leq |g_{n}^{(n-1)}| \leq |g_{n}^{(n-1)}| \). When this is no longer the case, the local character of the loading has reversed, and therefore the extreme gap value is overwritten by the last converged state \( g_{n-1}^{(n-1)} \). Using the new value of \( g_{n-1}^{(n-1)} \), Eqs. (49) and (50) become valid for computing the tangential traction, in analogy to (35). The interface then satisfies the reloading criteria so long as \( |g_{n}^{(n)}| \leq |g_{n}^{(n-1)}| \leq |g_{n}^{(n-1)}| \). When this criteria is no longer met, the interface reverts to the unloading condition. This procedure of updating \( g_{n-1}^{(n-1)} \) with respect to the extreme points in the loading–unloading cycles continues throughout the remainder of the simulation.

From this discussion, we observe that the gap values \( g_{n}^{(n-1)} \) and \( g_{n-1}^{(n-1)} \) are the state variables for the physics-based model. Finally, the various expressions for \( t_{n}^{(s)} \) given above are incorporated into equation (36) to arrive at the weak form containing the variationally-embedded interface friction model.

**Remark.** For complex structures containing multiple joints, the condition of load reversal locally at each joint may not be immediately evident at the initiation of the solution step. In this case, the first iteration can be made assuming each joint is under increasing load, and the incremental gap across each joint can be evaluated. Then each joint should be tested to see if the assumed incremental direction matches the sign of the gap function from the previous step. Those joints which experience a reversed increment should be updated using the unloading formula, and then the entire iteration should be solved again. A similar load reversal happens for joints that experience reloading after a partial unloading during a cyclic process.

**Remark.** In the discrete setting, the integrals shown in (36) are computed numerically by evaluating the arguments at specific integration points. Therefore, once the solution is obtained at the end of load step \( n \) – 1 or load branch \( s = 1 \), the values of the necessary displacements and tractions along the interface at each integration point are stored for subsequent use in the history-dependent formulas during the solution phase of load step \( n \).

### 4.3. Summary of proposed methods

To close this section, we present a summary of the methods proposed in this work in Table 2. The associated variational equa-

---

**Table 1**

<table>
<thead>
<tr>
<th>Algorithm for evaluating ( t_{n}^{(n)} ) from physics-based model.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Compute ( t_{n}^{(n)} = \frac{\phi_{n}^{(n)} + (c \mu_{t}/h)g_{n}^{(n)}}{C_0} )</td>
</tr>
<tr>
<td>2. Compute ( d ) from (47)</td>
</tr>
<tr>
<td>3. (a) If (</td>
</tr>
<tr>
<td>Compute ( t_{n}^{(n)} ) from (48)</td>
</tr>
<tr>
<td>(b) Otherwise:</td>
</tr>
<tr>
<td>Compute ( t_{n}^{(n)} ) from (46)</td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. (7)</td>
<td>Pure displacement form of the formulation for linear interface models</td>
</tr>
<tr>
<td>Eq. (17)</td>
<td>Mixed form of the formulation for linear interface models</td>
</tr>
<tr>
<td>Eq. (36)</td>
<td>Mixed form of the formulation for nonlinear, time-dependent interface models</td>
</tr>
</tbody>
</table>
tions are separated according to whether a single or mixed field treatment is employed and whether the method is appropriate for linear or nonlinear interface models.

**Remark.** We want to highlight that the ideas employed for the extension of the mixed field formulation to the corresponding iterative form (36) for nonlinear interface models can also be applied to the pure displacement-based formulation presented in Section 2.1. In that case, the discussion in this section can be directly appended to Eq. (7).

5. Numerical results

This section analyzes the performance of the stabilized interface formulation through a series of simulations to provide a comparison between the classical Coulomb and physics-based frictional models on a two-dimensional and three-dimensional sample problems. Linear triangular and quadrilateral elements are employed using equal-order interpolations for the displacement and pressure fields in Section 5.1 while linear tetrahedral and hexahedral pure-displacement elements are employed in Section 5.2. The bubble functions selected in [Masud et al., 2011] are used to represent the fine-scale fields in the mixed weak form. The value of the penalty parameter $\varepsilon = 10$ was used in all cases. Full numerical quadrature was employed in all the calculations; line segment interfacial integrals were evaluated using a three-point Gaussian quadrature scheme while surface interfacial integrals were evaluated using either a three-point or four-point rule for tetrahedral or hexahedral elements, respectively.

5.1. Two-dimensional lap joint

For the first series of numerical simulations, we selected a sample geometry representing small lap joints used in fretting experiments ([Eriten et al., 2011a]). Fig. 4 presents a diagram of the joint. The overlap at the interface is 16 mm long by 10 mm wide, and the two halves are 5 mm thick. The joint is considered fully-fixed at both ends. The material properties of steel are used, and plane stress conditions are assumed. A preload of 50 MPa is applied to represent the pretensioned bolts. Although the pressure distribution from the bolts is actually nonuniform, the pressure is assumed constant for simplicity. A tangential displacement of 5 microns is applied to the right end, and the coefficient of friction is taken as 0.3 at the interface. A plot of the general deformed configuration of the joint under this loading is shown in Fig. 5, where the displacements have been magnified two hundred times for visualization purposes, and the initial condition is given by the dashed lines.

In the simulations that follow, we compare the response of the lap joint predicted by the interface formulation incorporating the Coulomb model to that of the physics-based constitutive law. We also analyze the convergence of the numerical results upon mesh refinement as quantified by the error indicator described in Section 2.3. Three types of meshes were designed to illustrate the versatility of the computational framework: conforming and nonconforming meshes of quadrilaterals, and a composite nonconforming mesh including triangles and quadrilaterals. The mesh hierarchy employed for the convergence rate study is shown in Table 3. The coarsest meshes are shown in Fig. 6, and refined meshes are obtained via uniform bisection.

**Remark.** The meshes presented above were selected to conduct a convergence analysis and to obtain an estimate of the overall behavior of the joint. Local refinement near the ends of the joint overlap and near the edges of the pressure loading would likely be necessary to obtain greater accuracy in these zones; this observation is confirmed via the error estimates for the numerical results presented below.

5.1.1. Coulomb friction at interface

First, we present results from simulations of the lap joint employing the classical Coulomb model. Only results from the conforming mesh will be shown because of the similarity amongst the mesh types, which will be discussed in the following section. For the stick condition (45), we select a value for the stiffness parameter $\varepsilon_T = 4.0 \times 10^4$ N/mm$^2$. This value was selected numerically to make the results from classical Coulomb friction comparable to those for the physics-based model in the next section.

Contour plots of the displacement field are obtained on the medium mesh are shown in Fig. 7. These plots are smooth and free of oscillations at the interface. The compression of the joint under the applied pressure can be seen in Fig. 7(b). Partial slip of about half a micron can be deduced from the discontinuous contours along the joint in Fig. 7(a).

A plot of the relative displacement along the interface is given in Fig. 8. The normal gap $g_N$ is identically zero along the joint interface for the conforming mesh, and only very small deviations occur on the nonconforming and composite meshes. Almost no penetration is observed during any of the simulations, and this is attributed to the stabilization terms in the normal direction. The graph of the tangential gap $|g_T|$ is segmented to clearly exhibit the stick
and slip zones along the interface. The relative motion is symmetric about the centerline of the joint, in accordance with the loading and boundary conditions. The central zone of the interface under stick condition is about 4 mm wide, although the tangential gap is still almost 0.5 \( mm \) in this zone due to the relatively small value of \( e_T \). This relative motion more closely approximates the partial slip behavior of joints than the rigid response obtained from a larger value of \( e_T \).

To analyze the accuracy of these numerical results in solving the governing equations, we consult the value of the error indicator obtained on each mesh. A plot of the \( H^1 \) seminorm in each element of the medium mesh is presented in Fig. 9. In general, the fine-scale errors predict regions where the actual error in stresses and strains are relatively higher across the domain. Here, the errors appear to be higher at the ends of the joint and at the transition zones of the pressure on the top and bottom surfaces, which correlate with expected regions of reduced smoothness of the exact solution. These regions would be candidates for localized mesh refinement, while elements with low values of \( e \) can be considered to provide accurate approximations of the solution. In particular, much of the interface has about the same level of estimated error as most of the domain.

Plots of the error norms obtained from each mesh level are given in Fig. 10. The values have been normalized with respect to the corresponding displacement field norm computed on the finest mesh. These curves indicate the convergence of the numerical solution upon mesh refinement. The observed convergence rates of the \( L_2 \) norm and \( H^1 \) seminorm are slightly below the theoretical rates of 2 and 1, respectively; this can be attributed to reduced regularity of the solution due to boundary conditions and the joint overlap.

Remark. These error estimates are a key feature that distinguishes this stabilized formulation from standard frameworks that lack a built-in measure for assessing accuracy of solutions.

Finally, we conducted a quasi-static simulation using a cyclic applied displacement according to the method described in Section 4, and the resulting hysteresis loop is shown in Fig. 11. This plot tracks the value of the applied displacement \( \delta \) on the right end of the domain versus the tangential force reaction at the ends of the lap joint as obtained from the medium conforming mesh. A shift of about 300 N is observed between the unloading and reloading branches, which can be attributed to the relatively large portion of the interface under slip condition at the extremes of the loading cycle. The energy dissipated in one cycle is about 3.4 mJ, as calculated from the area inside the hysteresis loop. The closure of the loops implies that they are repeatable, and we have simulated five cycles without significant changes in the computed results.

5.1.2. Physics-based interface constitutive model

This section presents numerical results incorporating the physics-based model presented in Section 3. The values for the Greenwood-Williamson roughness and other surface parameters used in the simulations are given in Table 4. These surface roughness parameters were measured from an actual joint fabricated for fretting experiments (Eriten et al., 2011a).

The solutions from each of the three mesh types are compared as obtained from the variationally consistent formulation described in Section 4. Contour plots of the displacement field components for the medium level meshes are given in Figs. 12 and 13. The results obtained for the conforming and nonconforming mesh are nearly identical, with all major features of the solution captured on both meshes. The composite mesh also provides comparable results, with only minor variations observed in the top half of the joint between Fig. 13(a) and (c). These contour plots also appear qualitatively similar to the results from the classical Coulomb
friction simulation presented in Fig. 7. We emphasize that these results were obtained from the physics-based model that incorporates measurable surface properties rather than arbitrarily specified user parameters.

Remark. The uniformity of the results across the mesh types presented in Figs. 12 and 13 indicates that the proposed formulation produces numerical solutions of consistent quality independent of mesh conformity.

Next, we present plots of the normal and tangential gap functions along the joint interface in Fig. 14. The magnitude and variation of the tangential displacement are fairly consistent across the three mesh types. The graphs of the nonconforming meshes are not exactly symmetric; this is a consequence of the discretization because the discrete algebraic system is not exactly symmetric. However, the deviations are relatively insignificant compared to the overall trends obtained. There are also small oscillations in
the normal displacement along the interface; these spurious features are controlled by the stabilization terms and have been confirmed to decrease with mesh refinement.

The profile of the tangential displacement from the physics-based model agrees fairly well with the corresponding one shown in Fig. 8 for classical Coulomb friction, although some differences are apparent. Most noteworthy, the minimum value of $|g_y|$ shown in Fig. 14(a) is above 0.5 μm in the stick zone, and the size of the stick zone is much larger compared to Fig. 8. These differences can be attributed to the nonlinear force–displacement relationship.
Fig. 14. Interface displacement graphs: (a) conforming mesh; (b) nonconforming mesh; (c) composite mesh.

Fig. 15. Contour plot of $H^1$ seminorm of direct error $\tilde{e}$ (mm$^{3/2}$): (a) conforming mesh; (b) nonconforming mesh; (c) composite mesh.
obtained from the statistical summation approach described in Section 3. This relationship provides a smoother transition from stick to slip conditions compared to the abrupt change utilized in the classical Coulomb model.

An error analysis was also conducted for each mesh type. A plot of the element $H^1$ seminorm of $e$ is shown in Fig. 15 for the conforming, nonconforming, and composite meshes. The values for the conforming mesh shown in Fig. 15(a) are almost exactly the same as presented in Fig. 9. Therefore, the comments made previously concerning the errors caused by the sharp boundary conditions and the interface slip zones still apply. Comparing Fig. 15(a) to Fig. 15(b), the estimated error $e$ does not appear to have increased along the interface even in the presence of the nonconforming mesh. This is a testament to the stability of the proposed interface formulation.

Finally, we present results from convergence rate studies employing the error indicator $e$. Line plots of the $L_2$ norm and $H^1$ seminorm obtained from the series of meshes listed in Table 3 are shown in Fig. 16. These plots indicate that the estimated error for each mesh type is nearly the same, and the convergence rates are slightly below the optimal rates of 2 and 1 for the $L_2$ norm and $H^1$ seminorm, respectively.

Similar to the case of Coulomb friction, cyclic displacement-controlled loading was applied to simulate hysteretic behavior of the lap joint. The force–displacement curve is shown in Fig. 17 for the medium conforming mesh. Since the computed response is cyclic, we only report one cycle, although five contiguous loops were simulated. Compared to the Coulomb friction case, a maximum force differential of 500 N is observed during the cycles, and the total energy dissipation is about 3.9 mJ per cycle. These differences can be attributed to the smooth transition between partial slip and gross slip provided by the physics-based model compared to sharp elastic–plastic treatment of the Coulomb model. Again, we remark that a value for the elastic constant in the Coulomb model was assumed in order to enforce an agreement between the tangential forces computed from both constitutive laws. Alternatively, a different parametric value could be assigned to provide agreement in the total energy dissipation.

5.2. Three-dimensional double lap joint

The second series of simulations was conducted on a model of a double-bolted lap joint shown in Fig. 18(a). The joint consists of
two 1 cm thick, 10 cm long steel plates attached by 12 mm diameter bolts to lap plates that are 0.5 cm thick and 12 cm long. Each plate is 10 cm wide, and the bolts are evenly spaced at 5 cm on center, as can be seen from the plan view in Fig. 18(b). Rather than explicitly meshing the bolts and the holes in the plates, the bolt preload of 35 kN is applied as a uniform pressure over an area of 20 mm \times 20 mm which approximately corresponds to the area of the bolt head. The properties of steel given in the previous section, namely a Young's modulus of 20 GPa and a Poisson's ratio of 0.24, are assigned to the plates, and the friction coefficient is taken as 0.3.

A cyclic longitudinal displacement with maximum amplitude of \( \delta = 20 \mu \text{m} \) is applied uniformly along the edges of the 1 cm thick plates as shown in Fig. 18(a). Due to the symmetry of the problem domain, only one eighth of the joint is meshed, and appropriate symmetry conditions are applied. We consider two discretizations of the domain. The first is a structured conforming hexahedral mesh while the second contains tetrahedral elements in the
12 mm lap plates. The second mesh serves as an example of non-conforming meshes with different element types that can be encountered in practical applications when many connecting parts come into contact. Close agreement between the interfacial responses simulated by the two discrete models indicates that the interface formulation is insensitive to nonconforming meshes, as was shown for the two-dimensional problem in Section 5.1. In both cases, the meshes were generated from a template pattern of cubes with 1.67 mm edges, and the pure-displacement formulation corresponding to (7) was used for simplicity (see remark at the end of Section 4.3). The number of nodes in the mesh of the long plate and lap plate is 7564 and 4588, respectively. As a frame of reference, we assign the following coordinate system with its origin at the full-symmetry point as shown in Fig. 18(a) and (b): the $x$-axis is aligned with the cyclic load longitudinally, the $y$-axis is aligned with the preload in the thickness direction, and the $z$-axis is aligned with the transverse direction.

5.2.1. Coulomb friction

The mechanical response of the joint is first modeled using Coulomb friction, where the value of the stiffness parameter $\dot{\varepsilon}_T = 4.0 \times 10^7$ N/mm$^3$ is retained from the two-dimensional simulations. The applied displacement was increased from zero to the maximum value $\delta = 20 \mu m$ in steps of $2 \mu m$, and then it was cycled through one complete loop with a minimum value of $\delta = -20 \mu m$. The total shearing force on the joint is computed by summing the reaction forces at all nodes with prescribed cyclic displacement, and the force–displacement history during the cycle obtained from both meshes is plotted in Fig. 19. For this macroscopic quantity, the results from the tetrahedral and hexahedral element meshes are indistinguishable. The difference between the reaction forces at the maximum elongation at the start and end of the cycle was less than 1%, meaning that the hysteresis loops close as is physically expected. The ratio of the maximum total shear force across each interface to the total compressive force is about 0.23, or about 75% of the maximum capacity of the frictional interface. Finally, the energy dissipation per cycle was computed as 35 mJ.

Next, we plot displacement contours on the deformed configuration in Fig. 20 at the maximum load level $\delta = 20 \mu m$ to analyze more closely the spatial characteristics of the response. The deformations are magnified 200 times for visualization purposes. These contour plots are shown for half of the lap joint, where the solution in the other octants was obtained by reflecting the computed solution across the symmetry planes. From the distorted mesh lines in Fig. 20(a), we can observe the significant compression underneath the pretensioned bolts, which causes the lap plate to bend and raise the outer edges out of contact with the thicker plate. As will be mentioned below in the analysis of the contact stresses, only 27% of the interface remains in contact. The results from the non-conforming mesh are almost identical to the conforming mesh, highlighting a key strength of the numerical method. Only the upward deflection of the tips of the lap plate for the tetrahedral mesh is slightly less than observed from the hexahedral mesh. In Fig. 20, the $x$-component of displacement is noticeably discontinuous.
between the lap plates and the long plate. Also, the $u_z$ displacement contours in Fig. 20(c) and (d) illustrate the lateral contraction of the plates at the ends and in the center of the joint due to the Poisson effect. However, the deformations are more complex in the vicinity of the overlapping zone.

To examine the behavior close to the zone of contact, we construct a cut-away view of the portion of the joint in the $+x$-axis octant of the coordinate system and also remove half of the lap plate to expose the contact surface. Such regions are often inaccessible or difficult to measure during experiments, and thus computational techniques can serve a vital role in assessing the localized behavior of mechanical systems. The shear stress component $\sigma_{xy}$, which is the mechanism through which the applied load is transferred between the plates, is depicted in this cut-away view in Fig. 21; we again observe close agreement between the two mesh types even for this relatively coarse approximation with only three elements through the thickness. We remark that these contours are obtained through applying the nodal average post-processing technique to the element stress fields in each plate separately. From Fig. 21(a), the stress distribution is clearly non-uniform in the zone of contact. Upon removing the lap plate from view, the stress distribution resembles a "U" opening toward the applied displacement. From the visible portion of the lap plate, the stress field appears nearly continuous between the lap and long plates; also, it decays through the thickness of the lap plate to a small value underneath the applied preload. Due to the coarseness of the mesh, the stress is still apparent at the surface. This value is expected to decay upon refinement.

Finally, we present spatial distributions of the tangential traction and gap in the longitudinal direction in Fig. 22. These interface plots are taken from the perspective of an observer looking down at the long plate from above. Only the element surfaces within a 3 cm x 3 cm zone in the vicinity of active contact are shown, and zero values indicate regions that are not in contact. Although the results on the tetrahedral mesh exhibit slightly higher discontinu-

5.2.2. Physics-based friction model

The hysteresis simulation of the lap joint is now repeated using the physics-based model of Section 4.2. The same interface material properties listed in Table 4 are used for this study. Also, results obtained from both the conforming and nonconforming meshes are highlighted. The friction model is implemented in the x-direction...
tion aligned with the dominant loading, and elastic tangential springs with constant $c = 4.0 \times 10^4$ were employed in the transverse ($z$) direction to mimic the Coulomb friction response. The extreme values of the applied displacement were again taken as $\delta = \pm 20$ $\mu$m, and the time step size was adjusted variably between 2 $\mu$m and 10 $\mu$m.

Displacement contours obtained from both meshes at the maximum load level $\delta = 20$ $\mu$m are presented in Fig. 23. Again, the solution obtained from the conforming and nonconforming meshes are nearly indistinguishable. The longitudinal deformation of the joint is very similar to the results from the Coulomb friction simulation in Fig. 20. Slight variations in the characteristics of the transverse displacement near the contact zone can be noted between Fig. 20(c) and Fig. 23(c), where the magnitude of the deflections at the tips of the plates appears to be somewhat lower. However, the global trends are very much in agreement. Based upon the similarity in the results from the two meshes shown in Fig. 23 as well as the observations in the previous section, we are only presenting results for the conforming mesh in subsequent figures.

Next, a shear stress $\sigma_{xy}$ contour is shown on the cut-away view of the contact zone in Fig. 24. While the physics-based model produces many of the same features evident in the results from Coulomb friction in Fig. 21, a few distinct characteristics can be noted in the contact zone. First, the shear stress in the region closer to the applied displacement appears to be diffused over a larger area, and second the magnitude of the stress in the legs of the “U” appears to be reduced. These features are clearer in the contact traction and gap profiles shown in Fig. 25. Comparing the results in Fig. 25(a) to Fig. 22(a), the traction in the region $z \approx 32$ mm is lower for the physics-based friction compared to Coulomb friction but is comparable in the region $z \approx 17$ mm. Also, higher stresses are evident in the region $x \approx 43$ mm. Thus, the two models produce distinguishable results in the vicinity of the contact zone.

The force–displacement history produced by the cyclic loading is reported in Fig. 26. Similar values for the reaction forces at the extreme load levels are observed compared to those from the Coulomb model in Fig. 19. However, the area of the loop is noticeably smaller, and the energy dissipation per cycle is computed as only 15 mJ compared to the value of 35 mJ from Section 5.2.1. This discrepancy is likely attributable to the difference in the contact traction and gap profiles described above and illustrates the effects of interface modeling assumptions on the macroscopic response of the mechanical system model.

6. Conclusions

We have presented an application of a new method for interfacial modeling that utilizes concepts from discontinuous Galerkin (DG) methods employed within the context of a Variational Multiscale framework. The DG method (Masud et al., 2012) leads to interfacial flux terms that weakly enforce the continuity of the underlying fields and also provides a mechanism to embed friction models. A significant attribute of the method is an unbiased implementation of the interfaces that allows the deformation of the two interacting surfaces to conform to their local material and geometric features. The major contribution of this work is the embedding of two friction models into the DG interface formulation and developing appropriate time integration schemes for tracking the interfacial material response. The physics-based model proposed in Eriten et al. (2011a) is analyzed for a two-dimensional lap joint problem and used to calibrate the elastic stiffness parameter in the classical Coulomb model. Both models produce results that are in good agreement across a sequence of numerical tests with conforming and nonconforming meshes. A convergence rate study using the built-in error estimation module is also conducted to ensure that the simulation results are accurate and convergent, and the distribution of local error indicates that nonconforming meshes produce equally accurate results at the interface compared to conforming meshes. Following the two-dimensional analysis, a hysteresis study of a three-dimensional lap joint problem is conducted in which non-uniform contact interactions are present. The Coulomb friction model, which was previously investigated within the proposed formulation only for two-dimensional problems, produces physically meaningful results on coarse meshes and exhibits close agreement between results on conforming and nonconforming meshes. The physics-based friction model produces globally similar results to the Coulomb model but shows slightly different features in the contact stress and gap profiles. These numerical tests illustrate that the proposed interface formulation is capable of accurately capturing the discrete contact between non-matching meshes with different element types and accommodating complex friction models for simulating the fretting response of jointed structures.
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References


