A stabilized mixed finite element method for the incompressible shear-rate dependent non-Newtonian fluids: Variational Multiscale framework and consistent linearization

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A B S T R A C T

This paper presents a stabilized mixed finite element method for shear-rate dependent incompressible fluids. The viscosity of the fluid is considered a function of the second invariant of the rate-of-deformation tensor, thus making the shear-stress shear-strain relation nonlinear. The weak form of the generalized Navier–Stokes equations is cast in the Variational Multiscale (VMS) framework that leads to a two-level description of the problem. Consistent linearization of the fine-scale problem with respect to the fine-scale velocity field and the use of bubble functions to expand the fine-scale trial and test functions lead to an analytical expression for the fine-scale velocity along with a definition of the stabilization tensor $\tau$. The ensuing nonlinear stabilized form is presented and the consistent tangent tensor is derived. Numerical convergence of the proposed method on structured and unstructured meshes that are composed of linear triangles and bilinear quadrilaterals are presented. Shear-thinning and shear-thickening effects are investigated via the backward facing step problem and the effects of geometric parameters on the flow characteristics are highlighted. Time dependent features are investigated via the transient vortex-shedding problem and the accuracy and stability properties of the new method are shown.

1. Introduction

Flow of non-Newtonian fluids such as polymer solutions, molten polymers and biofluids shows shear-thinning or shear-thickening features that are not described by the linearly viscous Newtonian models with a constant value of viscosity [1]. The non-Newtonian flow properties are caused by the complex microstructures in these fluids. For example, in polymeric solutions and molten polymers, viscosity is a function of the molecular weight, concentration of the polymer, entanglement of the polymer chains and conformational changes in the polymer molecules [2]. Likewise, blood is a multi-component mixture of red blood cells (erythrocytes), white blood cells (leukocytes), platelets (thrombocytes) and the plasma. The viscosity of blood depends on the non-Newtonian rheological characteristics of its dominant components, erythrocytes and plasma [3]. Under high shearing the hydrodynamic forces in blood exceed the linking forces and the blood behaves as a highly shear-thinning liquid [4]. In general, this shear-rate dependent response of blood is dominant in the range $200$–$400 \text{ s}^{-1}$. A literature review reveals a wide class of constitutive models proposed for the shear-rate dependent fluids [5–9] (and references therein). Interested reader is directed to the works of Rajagopal and coworkers [1,10] wherein a unified approach to the modeling of blood and blood clots has been proposed.

For the numerical modeling of incompressible as well as compressible Navier–Stokes equations, the class of stabilized methods has become the methods of choice [11]. These methods date back to the works of Hughes and coworkers [12,13] on the development of the Streamline Upwind/Petrov–Galerkin (SUPG) method, and the Galerkin/Least-Square (GLS) method [14]. In the chronological history of stabilized methods a significant step was the notion of Variational Multiscale (VMS) framework [15,16] that turned out to be the basis for the development of the new generation of stabilized and multiscale finite element methods. This concept was employed by Masud and coworkers to develop stabilized/multiscale formulations for a variety of problem from mathematical physics [17–24]. Stabilized methods have also been developed in the form of the residual-free bubble methods by Brezzi and coworkers [25–29] and the unusual stabilized methods by Franca and coworkers [30,31]. Tezduyar et al. [32,33] and Masud and Hughes [34] extended stabilized methods to space–time finite element techniques. A good account on various developments in the area of stabilized methods is presented in a review article by Franca et al. [35].
This paper is an extension of our earlier works on the incompressible Navier–Stokes equations [21,22] to the generalized Navier–Stokes equations that are then integrated with the shear-rate dependent nonlinear constitutive models. Employing the Variational Multiscale decomposition we establish the coarse- and the fine-scale problems wherein adhesive as well as the diffusive terms are nonlinear. We employ the notion of consistent linearization of the nonlinear fine-scale problem with respect to the fine-scale velocity field to extract an analytical expression for fine-scales as well as for the stabilization tensor $\tau$. A variationally consistent projection of the fine-scales onto the coarse-scale space constitutes the new stabilized method. We also present the consistent tangent tensor emanating from the nonlinear stabilized form by linearizing with respect to the coarse-scale fields. The consistent tangent tensor yields great numerical efficiency in terms of faster convergence when used in conjunction with the Newton–Raphson scheme.

An outline of this paper is as follows. In Section 2 we discuss the shear-rate dependent constitutive models. The strong form and the standard weak form of the incompressible generalized Navier–Stokes equations are presented in Section 3. The derivation of the new stabilized method that is based on the Variational Multiscale (VMS) framework is presented in Section 4. The consistent tangent tensor for the nonlinear stabilized form is presented in Section 5. Section 6 presents the convergence-rate study for equal-order linear elements. Effects of the shear-rate dependent behavior are studied via the backward facing step problem for various Reynolds number flows. Time dependent features of the formulation are investigated via vortex-shedding around a circular cylinder. Conclusions are drawn in Section 7.

2. Stress–strain relation for shear-rate dependent fluids

This section presents constitutive equations for shear-thinning and shear-thickening fluids where shear-stress and shear-strain rates are related via nonlinear relations [5–9]. We introduce pressure as an independent field and split the stress tensor into volumetric stress and viscous stress components,

$$\sigma = -p I + \sigma_v,$$

where $\sigma$ is the stress tensor, $p$ is the pressure field or the volumetric stress, and $\sigma_v$ is the viscous stress which, in the case of shear-rate dependent fluids, is a nonlinear tensorial function:

$$\sigma_v = 2\eta(\gamma)\varepsilon_v,$$

$$\varepsilon_v$$ is the rate-of-deformation tensor which is defined as

$$\varepsilon_v := \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T).$$

For the case of non-Newtonian behavior of fluids, the viscosity field $\eta(\gamma)$ is a nonlinear function of the shear-rate $\gamma$ defined as:

$$\gamma := \sqrt{2\varepsilon_v : \varepsilon_v}.$$

Table 1 presents typical constitutive models for shear-rate dependent fluids. The coefficients $\mu_0$ and $\mu_\infty$ in these models represent the asymptotic values of the viscosity $\eta(\gamma)$ at zero and infinite shear-rates, respectively. For the Newtonian fluids we have $\eta(\gamma) \equiv \mu_0 = \mu_\infty = \mu$ for $\gamma$.

### Remark 1.

The developments for the new stabilized method presented in this paper are entirely general in that any of the constitutive models for the shear-rate dependent response of the fluid can be incorporated in the proposed method.

Fig. 1a shows viscosity as a function of shear-rate for various models. Newtonian fluids are represented via a constant viscosity. The power-law model displays a linear response with a constant slope that corresponds to the power of the model on the log–log scale. This model has an infinite viscosity at low shear-rates and zero viscosity at high shear-rates. The Carreau–Yasuda model possesses constant values of viscosity at both low and high ranges of shear-rate, and a varying viscosity in the intermediate range of shear-rate. Fig. 1b shows the shear-stress shear-rate relation for these models.

### Remark 2.

A good discussion on the stability of the at rest state for fluids with shear-rate dependent viscosity, and on the existence and regularity of the solution obtained via these models is presented in Málek et al. [36].

### Remark 3.

Based on the experimental investigation of human blood viscosity, Cho and Kensey [8] have proposed appropriate values of the constants to be used for the modeling of blood.
3. The incompressible generalized Navier–Stokes equations

3.1. Strong form of the problem

Let \( \Omega \subset \mathbb{R}^n \) be an open bounded region with piecewise smooth boundary \( \Gamma \). The number of space dimensions, \( n_{sd} \), is equal to 2 or 3. Strong form of the governing equations for an incompressible viscous fluid where viscous stress term may not necessarily be linear are given by the generalized Navier–Stokes equations written as follows:

\[
\begin{align*}
\rho \nu_t + \rho \nu \cdot \nabla \nu - \nabla \cdot \sigma_v(\nu) + \nabla p & = \rho f \quad \text{in } \Omega \times [0, T], \\
\nabla \cdot \nu & = 0 \quad \text{in } \Omega \times [0, T], \\
\nu & = g \quad \text{on } \Gamma_N \times [0, T], \\
\sigma_v \cdot n & = (\sigma_v(\nu) \cdot p) \cdot n = h \quad \text{on } \Gamma_h \times [0, T], \\
\nu(\boldsymbol{x}, 0) & = v_0 \quad \text{on } \Omega \times \{0\}.
\end{align*}
\]

These equations are supplemented by nonlinear viscous stress tensor \( \sigma_v \) defined as \( \sigma_v = 2\eta'(\gamma)\mathbf{I} \). In (5)–(9), \( \nu \) is the time derivative of the velocity field \( \nu \), \( p \) is the thermodynamic pressure, \( \rho \) is the density, \( f \) is the body force vector, \( g \) is the prescribed boundary velocities, \( h \) is the vector of the prescribed boundary tractions, \( v_0 \) is the prescribed initial velocity conditions, \( n \) is the unit normal vector on the boundary \( \Gamma \), and \( I \) is the first-order identity tensor. Eqs. (5)–(9) represent balance of momentum, the continuity equation, the Dirichlet and Neumann boundary conditions, and the initial condition, respectively.

3.2. The standard weak form

The appropriate spaces of weighting functions for the velocity and pressure fields are:

\[
\begin{align*}
\mathcal{W} &= \{ \mathbf{w} | \mathbf{w} \in (H^1(\Omega))^n, \mathbf{w} = 0 \text{ on } \Gamma_N \}, \\
\mathbb{Q} &= \{ q | q \in L_2(\Omega) \}.
\end{align*}
\]

The appropriate spaces of trial solutions \( S \) for the velocity field and \( \mathcal{P} \) for the pressure field are time-dependent spaces corresponding to \( \mathcal{W} \) and \( \mathbb{Q} \) presented in (10-a) and (10-b), respectively,

\[
\begin{align*}
S &= \{ \mathbf{v}(\cdot, t) | \mathbf{v}(\cdot, t) \in (H^1(\Omega))^n, \mathbf{v}(\cdot, t) = g \text{ on } \Gamma_N \times [0, T] \}, \\
\mathcal{P} &= \{ p(\cdot, t) | p(\cdot, t) \in L_2(\Omega) \}.
\end{align*}
\]

The standard weak form is: Find \( \mathbf{V} = (\mathbf{v}, p) \in S \times \mathcal{P} \) such that

\[
\begin{align*}
\rho(\mathbf{w}, \nu_t) + \rho(\mathbf{w}, \nu \cdot \nabla \nu) + (\nabla \mathbf{w}, \sigma_v(\nu)) - (\nabla \cdot \mathbf{w}, p) \\
= \rho(\mathbf{w}, \mathbf{f}) + (\mathbf{w}, \mathbf{h})_{\Gamma_h},
\end{align*}
\]

where \( \langle \cdot, \cdot \rangle = \int_{\Omega} \cdot \text{d}\Omega \) is the \( L_2(\Omega) \) inner product.

4. The Variational Multiscale method

The bounded domain \( \Omega \) is considered discretized into non-overlapping sub-regions \( \Omega_e \) (element domains) with boundaries \( \Gamma_e \), \( e = 1, 2, \ldots, n_{el} \) such that \( \Omega = \bigcup_{e=1}^{n_{el}} \Omega_e \). The union of element interiors and element boundaries is indicated by \( \Omega^e = \bigcup_{e=1}^{n_{el}} (\text{int } \Omega_e) \) and \( \Gamma^e = \bigcup_{e=1}^{n_{el}} \Gamma_e \), respectively.

We assume an overlapping sum decomposition of the velocity field into coarse- or resolvable-scales and fine- or subgrid-scales,

\[
\nu(\boldsymbol{x}, t) = \underbrace{\nu(\boldsymbol{x}, t)}_{\text{coarse-scale}} + \underbrace{\nu'(\boldsymbol{x}, t)}_{\text{fine-scale}}.
\]

We assume that \( \nu' \) is represented by piecewise polynomials of sufficiently high order, continuous in space but discontinuous in time. In particular, \( \nu' \) is assumed to be composed of piecewise constant-in-time functions leading to \( \nu(\boldsymbol{x}, t) = \nu(\boldsymbol{x}, t) + \nu'(\boldsymbol{x}) \). Consequently, \( \nu_0 = \nu_0 + \nu' = 0 \).

Likewise, we assume an overlapping sum decomposition of the weighting function into coarse- and fine-scale components indicated as \( \mathbf{w} \) and \( \mathbf{w}' \), respectively,

\[
\mathbf{w}(\boldsymbol{x}) = \underbrace{\mathbf{w}(\boldsymbol{x})}_{\text{coarse-scale}} + \underbrace{\mathbf{w}'(\boldsymbol{x})}_{\text{fine-scale}}.
\]

4.1. The Variational Multiscale formulation

Substituting the additively decomposed form of the velocity field and the weighting functions in Eqs. (12) and (13) leads to the following.

\[
\begin{align*}
\rho(\mathbf{w}, \nu_t) + \rho(\mathbf{w}, \nu \cdot \nabla \nu) + \rho(\mathbf{w}, \mathbf{v}' \cdot \nabla \mathbf{v}') + (\nabla \mathbf{w}, \mathbf{v}'(\mathbf{v} + \mathbf{v}')) - (\nabla \cdot \mathbf{w}, p) \\
= \rho(\mathbf{w}, \mathbf{f}) + (\mathbf{w}, \mathbf{h})_{\Gamma_h},
\end{align*}
\]

\[
(q, \nabla \cdot (\mathbf{v} + \mathbf{v}')) = 0,
\]

where the nonlinear viscous stress term is defined as

\[
\sigma_v(\mathbf{v} + \mathbf{v}') = 2\eta(\gamma')\mathbf{I}(\mathbf{v} + \mathbf{v}')
\]

and the rate-of-deformation tensor is written in terms of its coarse- and fine-scale components as follows

\[
\mathbf{e}(\mathbf{v} + \mathbf{v}') = \mathbf{e}(\mathbf{v}) + \mathbf{e}(\mathbf{v}') = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) + \frac{1}{2}(\nabla \mathbf{v}' + (\nabla \mathbf{v}')^T).
\]

We make a simplifying assumption that the nonlinear viscosity \( \eta(\gamma') \) is a function of the coarse-scale velocity field alone. Furthermore, for computational economy we assume that \( \gamma(\mathbf{v}) \) is piecewise constant over the element. Consequently, the nonlinear viscosity is based on the mean value of shear-rate \( \gamma \) that is derived from velocity field over the element, and is defined as

\[
\gamma = \frac{\int_{\Omega_e} \gamma(\mathbf{v}) \text{d}\Omega}{\int_{\Omega_e} \text{d}\Omega} = \frac{\int_{\Omega_e} \gamma(\mathbf{v}) \text{d}\Omega}{\text{meas}(\Omega^e)},
\]

where \( \text{meas}(\Omega^e) \) is a measure of the element size.

Remark 4. The assumption of \( \gamma(\mathbf{v}) \) being piecewise constant results in simplifying the resulting stabilized formulation and its consistent linearization, in addition to being a computationally economic option. A piecewise constant \( \gamma \) over the sum of element interiors \( \Omega^e \) is a first-order approximation to the nonlinearly varying \( \gamma \) field.

4.2. Coarse-scale sub-problem

Employing the linearity of the weighting function slot we can split (16) and (17) into coarse-scale and fine-scale sub-problems. These sub-problems can be written in residual forms as follows:

\[
\begin{align*}
R_1(\mathbf{w}, \mathbf{v}, \mathbf{v}', \mathbf{p}) & \overset{\text{def}}{=} \rho(\mathbf{w}, \nu_t) + \rho(\mathbf{w}, \mathbf{v} \cdot \nabla \mathbf{v}) + (\nabla \mathbf{w}, \mathbf{v}'(\mathbf{v} + \mathbf{v}')) \\
& + (\nabla \mathbf{w}, 2\eta(\gamma')\mathbf{I}(\mathbf{v} + \mathbf{v}')) - (\nabla \cdot \mathbf{w}, p) \\
& - \rho(\mathbf{w}, \mathbf{f}) - (\mathbf{w}, \mathbf{h})_{\Gamma_h} = 0,
\end{align*}
\]

\[
R_2(q, \mathbf{v}, \mathbf{v}') \overset{\text{def}}{=} (q, \nabla \cdot (\mathbf{v} + \mathbf{v}')) = 0.
\]

Eqs. (21) and (22) represent the weak forms of the balance of momentum and the continuity equations for the coarse-scale problem.
4.3. Fine-scale sub-problem

The weak form of the balance of momentum equation for the fine-scales is:

\[
R_3(\mathbf{w}, \mathbf{v}, \mathbf{v}'; p) \overset{def}{=} \rho(\mathbf{w}, \mathbf{v}; \mathbf{v}) + \rho(\mathbf{w}, (\mathbf{v} + \mathbf{v}') \cdot \nabla (\mathbf{v} + \mathbf{v}')) + (\nabla \mathbf{w}, 2\eta(\mathbf{v} \cdot \nabla \mathbf{v}')) - (\nabla \cdot \mathbf{w}, p) - \rho(\mathbf{w}, \mathbf{f}) = 0.
\] (23)

4.4. Linearization with respect to the fine-scale velocity field

Both coarse- and fine-scale problems are nonlinear with respect to the coarse velocity \( \mathbf{v} \) and fine velocity \( \mathbf{v}' \). We linearize these equations only with respect to the fine-scale velocity field. The linearization operators are defined as follows.

\[
\mathcal{L}(R_3(\mathbf{w}, \mathbf{v}, \mathbf{v}'; p) \overset{def}{=} \frac{d}{d\varepsilon} R_3(\mathbf{w}, \mathbf{v}, \mathbf{v}' + \varepsilon \mathbf{v}'), \bigg|_{\varepsilon=0}.
\] (24)

\[
\mathcal{L}(r_3(q, \mathbf{v}, \mathbf{v}')) \overset{def}{=} \frac{d}{d\varepsilon} r_3(q, \mathbf{v}, \mathbf{v}' + \varepsilon \mathbf{v}'), \bigg|_{\varepsilon=0}.
\] (25)

\[
\mathcal{L}(r_3(\mathbf{w}, \mathbf{v}, \mathbf{v}'; p) \overset{def}{=} \frac{d}{d\varepsilon} r_3(\mathbf{w}, \mathbf{v}, \mathbf{v}' + \varepsilon \mathbf{v}'), \bigg|_{\varepsilon=0}.
\] (26)

Applying the linearization operators (24)–(26) to the weak forms for momentum and continuity equations, we get the linearized formulations as follows:

\[
\rho(\mathbf{w}, \delta \mathbf{v} \cdot \nabla (\mathbf{v} + \mathbf{v}')) + \rho(\mathbf{w}, (\mathbf{v} + \mathbf{v}') \cdot \nabla \delta \mathbf{v}') + (\nabla \mathbf{w}, 2\eta(\mathbf{v} \cdot \nabla \mathbf{v}')) = -r_3(\mathbf{w}, \mathbf{v}, \mathbf{v}'),
\] (27)

\[
(q, \nabla \cdot \delta \mathbf{v}') = -r_3(q, \mathbf{v}, \mathbf{v}'),
\] (28)

\[
\rho(\mathbf{w}, \delta \mathbf{v} \cdot \nabla (\mathbf{v} + \mathbf{v}')) + \rho(\mathbf{w}, (\mathbf{v} + \mathbf{v}') \cdot \nabla \delta \mathbf{v}') + (\nabla \mathbf{w}, 2\eta(\mathbf{v} \cdot \nabla \mathbf{v}')) = -r_3(\mathbf{w}, \mathbf{v}, \mathbf{v}'),
\] (29)

where \( \eta(\mathbf{v}) \) is the value of viscosity that is based on the element-wise mean value of the coarse-scale based shear-rate \( \eta^e \) as defined in (20).

4.5. Solution of the fine-scale sub-problem

We rearrange (29) by keeping the \( \delta \mathbf{v} \) terms on the left hand side and taking all the other terms onto the right hand side

\[
\rho(\mathbf{w}, \delta \mathbf{v} \cdot \nabla \mathbf{v}) + \rho(\mathbf{w}, \mathbf{v} \cdot \nabla \mathbf{v}) + (\nabla \mathbf{w}, 2\eta(\mathbf{v} \cdot \nabla \mathbf{v})) = (\mathbf{w}', \mathbf{r}),
\] (30)

where \( \mathbf{r} = (\rho \mathbf{v} - \rho \mathbf{v} \cdot \nabla \mathbf{v} + 2\eta(\nabla \cdot \mathbf{v}) - \rho \mathbf{f}) \). It is important to note that \( \mathbf{r} \) represents the residual of the Euler–Lagrangian equations for the coarse-scale momentum balance equations. Because of the discretized nomenclature adopted herein, this residual is defined over the sum of element interiors.

4.5.1. Use of bubble functions to extract the fine-scale solution

We expand the fine-scale weighting and test functions via bubble functions \( b^e(\mathbf{x}) \).

\[
\mathbf{w} = b^e \gamma, \quad \delta \mathbf{v} = b^e \delta \beta,
\] (31)

where \( \gamma \) and \( \delta \beta \) are the coefficients for the weighting and the test functions, respectively.

We consider the first three terms on the left hand side of (30), and expand each term via bubble-functions.

First term:

\[
(\mathbf{w}', \delta \mathbf{v} \cdot \nabla \mathbf{v}) = \int (\mathbf{w}', (\delta \mathbf{v} \cdot \nabla \mathbf{v}) \cdot d\Omega = \gamma \cdot \int (b^e)^2 (\nabla \mathbf{v})^2 d\Omega \cdot \delta \beta.
\] (32)

Second term:

\[
(\mathbf{w}', \mathbf{v} \cdot \nabla (\delta \mathbf{v}') = \int \mathbf{w}' \cdot (\mathbf{v} \cdot \nabla (\delta \mathbf{v}')) d\Omega = \gamma \cdot \int b^e \mathbf{v} \cdot \nabla b^e d\Omega \cdot \delta \beta,
\] (33)

where I is the rank two identity tensor.

Third term:

\[
(\nabla \mathbf{w}', 2\eta(\mathbf{v} \cdot \nabla \mathbf{v}' = \gamma \cdot \int \eta(\nabla b^e \cdot \nabla b^e) d\Omega \cdot \delta \beta + \gamma \cdot \int b^e (\nabla b^e)^2 d\Omega \cdot \delta \beta.
\] (34)

Substituting (32)–(34) into (30) and solving the resulting equation we get the coefficients for the fine-scale velocity field \( \delta \mathbf{v} \). These coefficients are then used to reconstruct the fine-scale solution via Eq. (31).

If we assume the residual vector \( \mathbf{r} \) to be piecewise constant over the element interior, we can simplify the representation of the fine-scale velocity field:

\[
\delta \mathbf{v} = \tau \mathbf{r},
\] (36)

where \( \tau \) is the stabilization tensor. A piecewise constant \( \mathbf{r} \) amounts to considering the mean value of the residual over the element interior. Explicit definition of the stabilization tensor that emanates from the derivation presented above is as follows:

\[
\tau = b^e \left[ \rho \int (b^e)^2 (\nabla \mathbf{v})^2 d\Omega + \rho \int b^e \mathbf{v} \cdot \nabla b^e d\Omega + \int \eta(\nabla b^e)^2 d\Omega \right]^{-1} \int b^e \mathbf{r} d\Omega.
\] (37)

Remark 5. For numerical evaluation of \( \tau \) ideas from Masud and coworkers [21,22] can be employed.

4.6. The resulting coarse-scale sub-problem

In order to substitute the analytical expression for the fine-scale velocity field in the coarse-scale sub-problem, we reconsider (27) and rewrite the three terms on the left-hand side of (27) as follows:

\[
(\mathbf{w}, \delta \mathbf{v} \cdot \nabla \mathbf{v}) = (\nabla \mathbf{v} \cdot \mathbf{w}, \delta \mathbf{v}'.
\] (38)

\[
(\mathbf{w}, \mathbf{v} \cdot \nabla (\delta \mathbf{v}') = -((\nabla \cdot \mathbf{v}) \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w}, \delta \mathbf{v}'),
\] (39)

\[
(\nabla \mathbf{w}, 2\eta(\mathbf{v} \cdot \nabla \mathbf{v}')) = -((\eta(\nabla \cdot \mathbf{w}) + \Delta \mathbf{w}), \delta \mathbf{v}'),
\] (40)

Substituting the fine-scale solution \( \delta \mathbf{v} \) from (36) into the reformatted coarse-scale problem that is obtained by substituting (38)–(40) in (27), as well as in the continuity Eq. (28), we get the modified momentum and continuity equations:

\[
-((\rho - \nabla \cdot \mathbf{v}) \mathbf{w} + (\nabla \cdot \mathbf{v}) \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{w}, \delta \mathbf{v}') = -R_3(\mathbf{w}, \mathbf{v}, \mathbf{v}', \mathbf{p}),
\] (41)

\[
-((\nabla q, \mathbf{r} = -R_3(q, \mathbf{v}, \mathbf{v}').
\] (42)
4.7. The nonlinear stabilized form

The nonlinear stabilized form for the incompressible shear-rate dependent non-Newtonian fluids is derived from Eqs. (41) and (42).

$$\rho(w, v_t) + \rho(w, v \cdot \nabla v) + (\nabla w, 2\mu' \varepsilon(v)) - (\nabla \cdot w, p) + (q, \nabla \cdot v - (\chi, \tau_v)) - \rho(w, f) + (w, h),$$

(43)

where the weighting function for the stabilization term in Eq. (43) is defined as

$$\chi = \rho((-\nabla \cdot w + (\nabla \cdot v)w + v \cdot \nabla w) + \eta'((\nabla \cdot v)w + \Delta w) + \nabla q.$$  

(44)

Since the nonlinear stabilized form is completely expressed in terms of the coarse-scale fields, herein we drop the superposed bars from the coarse-scale velocity field.

The appropriate spaces for the pressure trial solutions and weighting functions for the stabilized form (43) are:

\[ \tilde{P} = \left\{ p, t \mid p, t \in L_2(\Omega), \nabla p \in L_2(\Omega)^{d \times d} \right\}, \]

\[ \tilde{Q} = \left\{ q \mid q \in L_2(\tilde{\Omega}), \nabla q \in (L_2(\tilde{\Omega}))^{d \times d} \right\}. \]

(45-a)  (45-b)

Let \( \varepsilon = \delta \times \tilde{P} \) and \( \gamma = \nabla \times \tilde{Q} \). Find \( V = \{ v, p \} \in \varepsilon \) such that, for all \( W = \{ w, q \} \in \gamma \),

\[ B_{stab}(W, V) = L_{stab}(W), \]

(46)

where \( B(\cdot, \cdot) \) is linear with respect to the first slot and is nonlinear with respect to the second slot

\[ B_{stab}(W, V) = B(W, V) + (\chi, \tau(v, p), \rho v \cdot \nabla w - 2\eta' \nabla \cdot \varepsilon(v) + \nabla q), \]

(47)

\[ L_{stab}(W) = L(W) + (\chi, \tau(f)), \]

(48)

\[ B(W, V) = \rho(w, v_t) + \rho(w, v \cdot \nabla v) + (\nabla w, 2\eta' \varepsilon(v)) - (\nabla \cdot w, p) + (q, \nabla \cdot v), \]

(49)

\[ L(W) = \rho(w, f) + (w, h), \]

(50)

where \( B_{stab}(W, V) \) and \( L_{stab}(W) \) are the operators for the nonlinear stabilized form, and \( B(W, V) \) and \( L(W) \) are the operators for the underlying Galerkin form as presented in (12) and (13).

Remark 6. In Eqs. (47) and (49) \( \eta' \equiv \eta'(\varepsilon) \) is the value of viscosity that is based on the element-wise mean value of the coarse-scale based shear-rate \( \varepsilon \) defined in Eq. (20).

5. The consistent tangent tensor

The Variational Multiscale method is based on the notion of an overlapping sum decomposition of the coarse- and fine-scale fields so that the solution to the resulting nonlinear stabilized form is obtained as an additive composition of these fields. To simplify the solution procedure for the fine-scale problem we assume a linear approximation of the fine-scale velocity \( \delta w \) and therefore the need to update the fine-scale velocity field is suppressed. This simplifying approximation leads to a simple definition of the stabilization tensor \( \tau \). The fine-scales are then variationally projected onto the coarse-scale space and they are manifested in the coarse-scale variational form via the stabilization terms. The resulting formulation is nonlinear with respect to the coarse-scale fields. Consequently, we need to linearize it and employ an iterative solution procedure for numerical solution of the problem.

Let us consider the nonlinear stabilized form (43) and rewrite it in a residual form

\[ R(w, v, p) = \rho(w, v_t) + \rho(w, v \cdot \nabla v) + (\nabla w, 2\eta' \varepsilon(v)) - (\nabla \cdot w, p) + (q, \nabla \cdot v - (\chi, \tau_v) - \rho(w, f) - (w, h). \]

(51)

We linearize (51) with respect to the coarse-scale fields. The linearization operator is defined as

\[ \mathcal{L}(R(w, v, p)) = \frac{d}{dc} \left\{ R(w, v + \delta v, p + \delta p) \right\} \bigg|_{c=0}. \]

(52)

Applying (52) to (51) leads to

\[ \rho(w, \delta v \cdot \nabla v) + \rho(w, v \cdot \delta v) + (\nabla w, 2\eta' \delta \varepsilon) + (\nabla \cdot w, \delta p) + (q, \nabla \cdot \delta v - (\chi, \tau \delta v) - R(w, v, p))^{(1)}, \]

(53)

where \( \delta v = v^{(1)}, \delta = e^{(1)} \), \( \delta e = e^{(1)} \) and \( (i) \) is an index for iteration. We can further simplify the fourth term as follows:

\[ \eta'(\delta \varepsilon) = \eta'(\varepsilon) \]

(54)

\[ \left( \frac{d}{dc} \right) \left( \frac{d\sqrt{\varepsilon}}{d\varepsilon} \right) = \left( \frac{d\sqrt{2\varepsilon}}{d\varepsilon} \right) = \left( \frac{1}{2} \right) = \frac{1}{2}. \]

(55)

\[ \mathcal{L}(e : e) = 2\varepsilon : e \delta \varepsilon = 2\varepsilon : e \delta e. \]

(56)

The linearized residual \( \delta r \) in the last term on left hand side of (53) is defined as

\[ \delta r = \frac{d}{dc} \left\{ r(v + \delta v, p + \delta p) \right\} \bigg|_{c=0} = -\rho \delta v \cdot \nabla v - \rho v \cdot \nabla (\delta v) + 2\eta' \nabla \cdot \delta e + 4\eta' \nabla \cdot \delta \varepsilon \]

(57)

We can now write the stabilized form with the consistent tangent tensor as

\[ \rho(w, \delta v \cdot \nabla v) + \rho(w, v \cdot \delta v) + (\nabla w, 2\eta' \delta \varepsilon) + (\nabla \cdot w, \delta p) + (q, \nabla \cdot \delta v - (\chi, \tau \delta v) + R_1(w, v, p), \]

(58)

where the left hand side is the consistent tangent tensor written in terms of the incremental solution fields \( \delta v, \delta p \). The right hand side is the residual vector at (i)th iteration and is composed of three parts where \( R_1(w, v, p), R_2(q, v), \) and \( (\chi, \tau \delta v) \) are the residuals from the momentum balance Eq. (21), the continuity Eq. (22), and the stabilization terms, respectively.

6. Numerical results

This section presents numerical testing of the proposed method. We have employed the Carreau–Yasuda model in Sections 6.1 and 6.2 and the Power law model in Section 6.3 as the representative constitutive models for shear-rate dependent fluids, respectively. Values of the constants for the Carreau–Yasuda model are given in Table 2. Fig. 2 shows equal order pressure–velocity elements comprising linear triangles and bilinear quadrilaterals. Standard quadratic bubbles for triangles and quadrilaterals are employed for the evaluation of the stability parameter \( r \) given in (37). Full Gauss quadrature rules are employed for numerical integration. In the numerical implementation of the Newton–Raphson method, nonlinear iterations are carried out on the coarse-scales while the fine-scales are treated as linear during the iterations for the
coarse-scales. We first present the rate of convergence study for uniform meshes, distorted meshes and composite meshes that are composed of triangles and quadrilaterals in the same computational domain. Effect of geometric parameters on the shear-rate dependent response is investigated via the backward facing step problem. Transient features of the method are studied via vortex-shedding from a circular cylinder.

6.1. Rate of convergence study

Rate of convergence study is a numerical check of the mathematical consistency and stability of discrete formulations. Standard convergence estimates are available for the convergence of the pressure and velocity fields and are expressed in terms of the order of the interpolation polynomials employed. In addition to the advection terms that are nonlinear, in the present formulation viscosity is a function of the velocity field which renders the diffusion term also nonlinear. We assume the following velocity and pressure profiles for the exact solution

\[
\begin{align*}
v(x,y) &= \frac{V_0}{(x^2 + y^2)^{1/2}} \cos(2\pi y) \sin(2\pi x), \\
p(x,y) &= \frac{\rho g}{4} \left( \frac{200}{\rho g (x^2 + y^2)} + \frac{40}{100 (x^2 + y^2)} \right) \sin(2\pi x) \cos(2\pi y),
\end{align*}
\]

where the bi-unit domain is selected such that \(-0.5 \leq x, y \leq 0.5\). It is important to note that the assumed velocity field satisfies the incompressibility condition given by Eq. (6). Substituting the expressions for the velocity and pressure fields in the governing Eq. (5) yields the body force vector, given as:

\[
f = \begin{bmatrix}
-xe^{10r^2} - \frac{1}{10} (200\eta/(5r^2 + 1))xe^{10r^2} + \frac{\rho g}{40 + 100r^2}xe^{10r^2} - 2\pi e^{5/4} \cos(2\pi x) \sin(2\pi y) \\
-y e^{10r^2} + \frac{1}{10} (200\eta/(5r^2 + 1))ye^{10r^2} + \frac{\rho g}{40 + 100r^2}ye^{10r^2} + 2\pi e^{5/4} \sin(2\pi x) \cos(2\pi y)
\end{bmatrix},
\]

where \(r^2 = x^2 + y^2\). Fig. 3a and b presents the components of the derived body force vector on the bi-unit square \((-0.5 \leq x, y \leq 0.5)\). This body force is then employed to run the discrete problem.

Fig. 4a presents the magnitude of the exact velocity field. Fig. 4b presents the profile of the exact pressure field given by Eq. (60).
Fig. 4c presents the profile of the shear-rate field $\dot{\gamma}$ which is defined in Eq. (4). Fig. 4d shows the exact viscosity field $\eta(\dot{\gamma})$. This section is further divided into four sub-sections.

1. Convergence rates for regular structured meshes.
2. Convergence rates for distorted and graded meshes.
3. Convergence rates for heterogeneous meshes where elements with rapidly varying topology exist next to each other.
4. Convergence rates for composite meshes where triangles and quadrilaterals are seamlessly embedded in the same computational grid.

We are dealing with a nonlinear finite element formulation where nonlinearity is engendered by both the advective and the diffusive terms. In order to get a converged solution on a given mesh, the nonlinear problem is run until numerical convergence is attained. Acceptable tolerance to reach convergence in nonlinear iterations is set equal to $10^{-16}$. Once the converged solution is attained, the error norms of the computed solution with respect to the exact solution are computed. We report the convergence rates in terms of the $L_2$-norm of the velocity field and $H^1$-seminorm of the pressure field. Also reported are the rates for $L_2\text{div}\cdot \mathbf{v}$ and $L_2p$ norms.

Remark 7. Test cases 2–4 are designed to highlight typical features in meshes that are employed for the analysis of problems encountered in engineering applications.

6.1.1. Convergence study for structured meshes

Fig. 5 shows structured triangular and quadrilateral meshes. In each case subsequent meshes are designed such that the coarser
discretization is fully embedded in the refined discretization. Triangular meshes are generated by bisecting the quadrilateral meshes such that the number of degrees of freedom is same between the two mesh types. Figs. 6a and 6b show the rates for linear triangles and bilinear quadrilaterals, respectively. In all the cases optimal convergence rates are attained in the norms considered.

6.1.2. Convergence study for distorted meshes (Case A)

The graded meshes are created by uniformly skewing the structured meshes as shown in Fig. 7. Figs. 8a and 8b present the convergence rates that confirm the theoretically predicted rates for the element types considered.

6.1.3. Convergence study for distorted meshes (Case B)

The spatial discretizations considered in this section are highly heterogeneous and are composed of unequal size elements next to each other (see Fig. 9). Such discretizations are typically encountered in industrial strength problems where mesh refinements are employed in the localized regions of the computational domain either to generate a better representation of the geometry, or to model the high variation in the computed solutions. Figs. 10a
and 10b show the computed convergence rates and in all the cases optimal rates are obtained for the velocity field. Optimal rates are also attained for the pressure field for the quadrilateral element, however the triangular element shows slight degradation in the rates of convergence for such high heterogeneity in the spatial discretization.

6.1.4. Convergence study for composite meshes

This subsection presents convergence of the fields on composite meshes where different element types are seamlessly embedded in the computational grids. It shows that off-the-shelf mesh generators can be employed to glue different element types in the discretization of complex geometries and the proposed stabilized method can be used to yield stable and convergent solutions. This is one of the major attributes of the proposed formulation and is of great practical importance in difficult geometric configurations.

The first set of composite meshes is composed of structured sub-meshes wherein linear triangles and bilinear quadrilaterals are glued together. Fig. 11 shows one typical mesh, and refined meshes are obtained by further sub-dividing the elements in their respective subdomains. Fig. 12 shows the convergence rates for the velocity and pressure fields, and nearly optimal convergence is attained in the norms considered.

Figs. 13 and 14 show the graded composite mesh and the convergence rates for the pressure and velocity fields, respectively. Fig. 15 shows a composite mesh that is composed of slanted layers of triangles and quadrilaterals. Finer meshes are obtained by systematically sub-dividing triangles and quadrilaterals. The computed pressure field is slightly sub-optimal, however as shown in Fig. 16, full convergence rates are attained for the velocity field in the norms considered.

Fig. 8b. Equal-order bilinear quads.

Fig. 10a. Equal-order linear triangles.

Fig. 10b. Equal-order bilinear quads.

Fig. 9. Highly heterogeneous meshes employed in the study.

Fig. 11. Regular composite meshes employed in the study.
Fig. 17 shows composite meshes that are composed of triangles and quads, and are conceivably the most heterogeneous meshes. Fig. 18 shows the computed convergence rates.

6.2. The backward facing step problem

Fig. 19 shows the schematic diagram of the backward facing step problem used to investigate the characteristics of shear-thinning and shear-thickening fluids. A constant velocity profile along
the $x$-axis is imposed at the inflow boundary while traction free boundary condition along the $x$-axis is applied at the outflow boundary (i.e., $\sigma_{xx} n_x + \sigma_{xy} n_y = k_x = 0$). Velocity profile along $y$-axis at both the inflow and the outflow boundaries is set to zero. No slip boundary conditions are applied along the bounding top and bottom surfaces.

The behavior of Newtonian fluids is expressed in terms of the Reynolds number which is a function of the ratio of advective and diffusive forces and is a non-dimensional parameter

$$Re = \frac{\rho u_0 (2h)}{\mu_{\text{min}}}.$$  \hspace{1cm} (62)

**Remark 8.** For the shear-thinning fluid $\mu_{\text{min}} = \mu_\infty$, and for the shear-thickening fluid $\mu_{\text{min}} = \mu_0$.

However for the shear-rate dependent fluids, in addition to the Reynolds number we need information on the shear-rate that affects the local viscosity field in the fluid. The local shear-rate in these fluids is affected by the geometric parameters of the spatial configuration. In other words, flow features of a shear-rate fluid in tubes of different diameters vary considerably even when the Reynolds number is kept the same. In order to quantify the flow features of such fluids we define another parameter, namely, the normal inflow shear-rate $\gamma_0$ defined as

$$\gamma_0 = \frac{2u_0}{h|_{Re=1}}, \quad \gamma_{\text{inflow}} = Re \cdot \gamma_0.$$  \hspace{1cm} (63)

This definition of the normal inflow shear-rate $\gamma_0$ is based on the wall shear-rate of the fully developed inflow velocity profile for the Newtonian fluids (i.e. parabolic shape) corresponding to Reynolds number $Re = 1$. Accordingly, for a given geometry, as the Reynolds number is increased, there is a corresponding quantifiable change in the inflow shear-rate $\gamma_{\text{inflow}}$.

**Remark 9.** The flow features for a Newtonian fluid at a given Reynolds number in channels of various diameters are essentially unaltered. However the features in the flow of shear-rate dependent fluids vary substantially.

**Remark 10.** For flow through parallel plates $\alpha = 6$, and for flow through a tube of circular cross section, $\alpha = 8$ in Eq. (63).

**Remark 11.** Let $\beta$ represent the ratio of diameter of the larger and smaller tubes. For a given $Re$, the shear-rate at inflow for the smaller diameter tube increases by a factor $\beta^2$.

**Remark 12.** The definition of inflow shear-rate (63) and the data presented in Table 3 reveal that in shear-rate dependent fluids, geometric length scale is coupled with the Reynolds number. If the inflow shear-rate $\gamma_{\text{inflow}}$ is to be kept constant for the three geometries in Table 3, Eq. (62) reveals that the corresponding $Re$ has to be reduced by a factor $\beta^2$.

### Table 3

<table>
<thead>
<tr>
<th>Geometry type</th>
<th>$H$ (mm)</th>
<th>$\gamma_0$ (s$^{-1}$)</th>
<th>$\gamma_{\text{inflow}}$ (s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Re = 50$</td>
<td>$\gamma_0$</td>
<td>0.003906</td>
<td>0.1953</td>
</tr>
<tr>
<td>$Re = 500$</td>
<td>$\gamma_0$</td>
<td>0.3906</td>
<td>19.53</td>
</tr>
<tr>
<td>$Re = 4500$</td>
<td>$\gamma_0$</td>
<td>39.06</td>
<td>1953</td>
</tr>
</tbody>
</table>

**Fig. 19.** Schematic diagram of the backward facing step problem.

**Fig. 20.** Length of recirculation zone as a function of $Re$.

**Fig. 21.** Wall shear-stress along the top surface.

### 6.2.1. Validation of the numerical results

Using the finite volume method Choi and Barakat [37] have presented numerical results for the backward facing step problem for a shear-thinning fluid. We employ their results to validate our method for a Newtonian fluid and a shear-thinning fluid. Geometric parameters for this test case are:

$h = 5.2$ mm, $s = 0.9423h$, $L_1 = 15h$, $L_2 = 30h$. 

---

The parameters employed in our study are same as the ones used in [37] and are given in Table 2. The backward facing step is located at \( x = 15h \). Fig. 20 shows normalized lengths of the recirculation zone for different Reynolds numbers and Fig. 21 presents the comparison of wall shear-stresses. A good comparison is attained in both the cases.

### 6.2.2. Effect of geometric parameters on the response of shear-rate dependent fluids

In order to study the effect of geometric parameters on the flow of shear-rate dependent fluids, three geometric configurations, presented in Table 3, are tested for a range of Reynolds numbers, where \( 50 \leq Re \leq 4500 \). In this range of Reynolds numbers, the inflow shear-rates for geometry G1 produce dynamically varying viscosity, while for geometry G3 these shear-rates generate asymptotic viscosity. The geometry G2 is an intermediate case between G1 and G3.

Figs. 22a and 22b show normalized lengths of the recirculation zones for the various Reynolds number flows and for different geometric configurations for shear-thinning and shear-thickening fluids, respectively. For shear-thinning fluids, G1 shows marked difference from the response of Newtonian fluids due to the dynamically varying viscosity of the fluid. Geometry G2 shows appreciable difference at low \( Re \), while the lengths of recirculation zones concur with that of the Newtonian fluids at high \( Re \) values. Plots for G3 are almost identical to that for the Newtonian fluids because for this geometric configuration the shear-rate is uniformly high all through the domain. This renders the asymptotic viscosity of the shear-thinning fluids approach the viscosity of the Newtonian fluids.

Normalized lengths for shear-thickening fluids show substantially different response as compared with the Newtonian fluids over almost the entire range of Reynolds numbers except the very low \( Re \) values. It is because the asymptotic viscosity of the shear-thickening fluids matches the viscosity of the Newtonian fluid at low shear-rates.

Figs. 23–25 show the velocity magnitude contours with superposed streamline for the shear-thinning and shear-thickening fluids. The Carreau–Yasuda model is employed as the representative constitutive model for the shear-rate dependent fluid. The velocity field for the Newtonian fluid (with a constant viscosity \( \mu = 0.00345 \text{ Pa s} \)) is also presented in Figs. 23a, 24a and 25a to highlight the difference from the two types of shear-rate dependent fluids. While the Newtonian fluid shows qualitatively similar contours and streamlines for the three geometric configurations, the shear-rate dependent fluids show a significant geometry dependent response.

Figs. 23a–g presents plots for \( Re = 50 \) for the three geometries G1, G2 and G3. For the wide channel G1 (\( \gamma_{\text{inflow}} = 1953 \text{ s}^{-1} \)), the shear-thickening fluids behave like the Newtonian fluids because the low shear-rates yield a viscosity field that is asymptotically close to the viscosity of the Newtonian fluids. For the narrow channel G3 (\( \gamma_{\text{inflow}} = 1953 \text{ s}^{-1} \)), shear-rates are high enough for the viscosity of the shear-thinning fluids to asymptote to the viscosity of the Newtonian fluids and therefore Fig. 23d resembles Fig. 23a.

On the other hand, for shear-thickening fluids as the shear-rate evolves either via an increase in \( Re \) or by a reduction in the geometric length scale (Table 3), the lengths of the recirculation zones reduce. It is because the higher local shear-rates near recirculation zones cause an increase in the viscosity of the shear-thickening fluids, thereby adding to the dissipation effects.

Figs. 24a–g presents the velocity magnitude contours with superposed streamlines for \( Re = 500 \). At \( Re = 500 \) the relatively higher shear-rate range shifts the viscosity field toward the asymptotic viscosity \( \mu_{\infty} \). In addition, the reduction in the geometric length parameter for the intermediate channel G2 and the narrow channel G3 results in higher shear-rates that substantially increase the lengths of the recirculation zones from G1 (\( \gamma_{\text{inflow}} = 1953 \text{ s}^{-1} \)) to G3 (\( \gamma_{\text{inflow}} = 19530 \text{ s}^{-1} \)). The shear-thickening fluids on the contrary show decreasing lengths of the recirculation zones, as presented in Fig. 24e–g.

Figs. 25a–e presents the velocity magnitude contours with superposed streamlines at \( Re = 4500 \). Except for the wide channel G1 with low inflow shear rate (\( \gamma_{\text{inflow}} = 17.577 \text{ s}^{-1} \)), viscosity fields for the intermediate channel G2 and the narrow channel G3 asymptote to that of the Newtonian fluid, and therefore show similar velocity contours and streamlines as the Newtonian plot.

**Remark 13.** Response of Newtonian fluids does not depend on the normal inflow shear-rates \( \gamma_0 \), rather it only depend on the Reynolds numbers. Velocity fields of Newtonian fluids in Figs. 23–25 show qualitatively similar contours for the given Reynolds numbers with the only difference in the magnitude of the computed velocity fields.

Figs. 26 and 27 present viscosity line plots for the shear-thinning fluids and the shear-thickening fluids at two locations downstream from the backward facing step. The first of the two locations is at
$x = h$ that always lies in the recirculation zone for all the test cases. Fig. 26 shows the viscosity plots for an intermediate range of shear-rates. As can be seen, the response is a function of $Re$ as well as of the geometric parameters. $G_1$ for $Re = 50$ and $Re = 4500$ and $G_2$ for all Reynolds numbers show an order of magnitude variation in viscosity. $G_1$ at $Re = 50$ shows the initial viscosity and $G_3$ for all Reynolds numbers shows the asymptotic viscosity. Due to the presence of the recirculation zone at $x = h$, viscosity line plots show two
Fig. 25. Streamlines near the step at $Re = 4500$.

Fig. 26a. Shear-thinning viscosity profiles at $x = h$.

Fig. 26b. Shear-thickening viscosity profiles at $x = h$. 
peaks for the intermediate range of shear-rates. One peak lies in the middle of the main stream flow and the other lies in the middle of the recirculation zone.

Fig. 27 shows viscosity plots at $x = 100h$ where the flow is fully developed. An order of magnitude variation in the value of viscosity for an intermediate range of shear-rates is plotted along y-axis, and a dominant single peak can be seen in Fig. 27.

6.2.3. Normalized wall shear-stress (WSS) downstream of the backward facing step

This section investigates the effects of geometric length parameter on the wall shear-stresses for the shear-rate dependent fluids. The WSS along the top surface of the channel is normalized by the reference shear-stress $\tau_{ref}$ defined as the inflow shear rate given in (63) multiplied by the minimum viscosity of the fluid.

![Fig. 27a. Shear-thinning viscosity profiles at x = 100h.](image1)

![Fig. 27b. Shear-thickening viscosity profiles at x = 100h.](image2)

![Fig. 28a. Normalized wall shear-stresses for Re = 50.](image3)
\[ \tau_{ref} = \mu_{\text{min,inflow}}. \]  

Fig. 28a shows the normalized wall shear-stresses at \( Re = 50 \). The shear-thinning fluid for the narrow channel G3 shows similar stress profile as that of the Newtonian fluids. Likewise, the shear-thickening fluid for the wide channel G1 shows similar plots as the Newtonian fluids because the initial viscosity of the shear-thickening fluid is the same as the Newtonian viscosity.

Fig. 28b presents the normalized wall shear-stresses at \( Re = 500 \). Since the shear-rates are larger than the ones at \( Re = 50 \), all shear-thickening fluids show deviation from the Newtonian plots. The shear-thinning fluid for medium and narrow channel show WSS close to the Newtonian WSS.

Fig. 28c shows the normalized wall shear-stresses at \( Re = 4500 \). The shear-thinning fluids for G2 and G3 behave like the Newtonian fluids because of the asymptotic viscosity approaching the Newtonian viscosity. Due to high shear-rate the shear-thickening fluids for G2 and G3 also show nearly similar plots.

6.3. The transient vortex-shedding problem

In this section we investigate the time dependent attributes of the proposed method. It is important to note that for the new stabilized method presented in Section 4.7, the consistent mass matrix is composed of two parts.

<table>
<thead>
<tr>
<th>Mesh type</th>
<th>( H/D )</th>
<th>( L_w/D )</th>
<th>( L_d/D )</th>
<th>( N_{node} )</th>
<th>( N_c )</th>
<th>( \delta/D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>90</td>
<td>20</td>
<td>80</td>
<td>11,136</td>
<td>80</td>
<td>0.04</td>
</tr>
<tr>
<td>M2</td>
<td>90</td>
<td>20</td>
<td>80</td>
<td>26,582</td>
<td>158</td>
<td>0.02</td>
</tr>
<tr>
<td>M3</td>
<td>90</td>
<td>20</td>
<td>80</td>
<td>34,506</td>
<td>314</td>
<td>0.01</td>
</tr>
<tr>
<td>Patnana et al. [38]</td>
<td>90</td>
<td>20</td>
<td>80</td>
<td>219,610</td>
<td>240</td>
<td>0.02</td>
</tr>
</tbody>
</table>
Consistent mass \(= \rho \langle \mathbf{w} \cdot \mathbf{v} \rangle + \rho \langle \mathbf{z} \cdot \mathbf{v} \rangle \). \hspace{1cm} (65)

The first term appears from the underlying Galerkin part of the formulation, and the second term emanates because of the projection of the fine-scales onto the coarse-scale space, as given in Eq. (47).

We have implemented this consistent mass matrix in our implicit time integration scheme, and have employed full numerical quadrature for the evaluation of the mass matrices.

The test case considered is two-dimensional vortex-shedding from a circular cylinder. The shear-rate dependent constitutive model employed here is the power-law model with \(n = 0.4\). We investigate the transient vortex-shedding behavior of the shear-thinning fluid at \(Re = 100\) and compare our numerical results with the results presented in Patnana et al. [38], obtained via the finite volume method. In this numerical study, the Reynolds number for the power-law fluid is defined as follows:

\[
Re = \frac{\rho u_0^2 \cdot n D^2}{\mu},
\]  

(66)

where \(\rho\) is the density of the fluid, \(u_0\) is the inflow velocity, \(D\) is the diameter of the cylinder, \(n\) is the power index of the power-law model and \(\mu\) is the viscosity parameter of the power-law model as described in Table 1.

For time integration, the second-order accurate BDF2 method is used. Time increment \(\Delta t = 0.1\) is used for the implicit time integration all through this section.
Fig. 29 presents schematic diagram of the computational domain. A constant horizontal velocity is applied at the inflow, while traction free boundary condition is applied at the outflow. No slip boundary conditions are applied on surface of the cylinder and transverse component of the velocity field is set equal to zero along the top and bottom walls to represent infinitely continuous domain along the y-axis.

Table 4 provides the mesh parameters for the three successively refined meshes employed in the present study. Also presented is the geometric description of one mesh used in Patnana et al. [38]. The simulation results from [38] are employed for comparison purpose in the present study. \( H \) is height of the computational domain, \( L_u \) is the upstream length, \( L_d \) is the downstream length, \( N_{node} \) is the number of nodes in the entire domain, \( N_i \) is the number of nodes on the surface of the cylinder, and \( \delta \) is the length dimension of the elements normal to the surface of the cylinder. Fig. 30a and b shows the close up view and the zoomed view of the coarse mesh M1, respectively. Refined mesh M2 and M3 were generated via successive refinements.

Remark 15. It is important to note that even the finest mesh used in the present study only contains 15.7% nodes as compared to the one used in Patnana et al. [38]. Accordingly, the proposed method results in over two orders of magnitude reduction in the cost of computation for the vortex shedding problem presented here.

Figs. 31a and 31b present the drag coefficient \( C_D \) and the lift coefficient \( C_L \) of the cylinder in the fully developed flow regime for the mesh M3. These coefficients are defined as

\[
C_D = \frac{\int (p - \sigma \cdot n) \cdot dS / (1/2) \rho u_0^2 D}{(1/2) \rho u_0^2 D},
\]

\[
C_L = \frac{\int (p - \sigma \cdot n) \cdot dS / (1/2) \rho u_0^2 D}{(1/2) \rho u_0^2 D}.
\]

Figs. 32a–c present the surface pressure coefficients for meshes M1–M3, respectively. The surface pressure coefficient \( C_p \) is defined as

\[
C_p = \frac{(p(h) - p_0)/(1/2) \rho u_0^2)}{(1/2) \rho u_0^2},
\]

where \( p(h) \) is the pressure on the surface of the cylinder at angle \( h \) and \( p_0 \) is the pressure at the outflow.

Fig. 33a–d presents contours of the velocity field with superimposed stationary streamlines for one representative cycle. Fig. 34a and b presents pressure contours half cycle apart. Likewise Fig. 35a and b presents contours half cycle apart of the shear-rate field that determines the local viscosity field for the shear-rate dependent...
fluid. High shear-rate zones are observed around the cylinder. The contours of the shear-rate field are qualitatively similar to the viscosity contours presented in Fig. 36a and b. Based on the shear-rate field, viscosity near the cylinder shows shear-thinning effect while viscosity in the free stream is high.

7. Conclusions

We have presented a new mixed finite element method for shear-rate dependent incompressible fluids. In these fluids the viscosity is considered a function of the second invariant of the
rate-of-deformation tensor, thus making the diffusive term also nonlinear. The stabilized form for the shear-rate dependent fluids is derived via the Variational Multiscale (VMS) method. The idea of consistent linearization of the fine-scale problem only with respect to the fine-scale field simplifies the subgrid-scale modeling of the problem. A simplifying assumption that nonlinear viscosity is only a function of the coarse-scale velocity field leads to considerable simplifications in the structure of the stabilization terms as well as in the derivation of the consistent tangent tensor. Convergence rate studies are presented on a variety of mesh types that corroborate the theoretical convergence rates for the norms considered. The effects of geometric parameters on the flow characteristics of shear-rate dependent fluids are highlighted via the backward facing step problem. Geometric parameters are varied to uniformly change the dimensions without altering the geometric shape of the backward facing step. It is shown that for a given Reynolds number, the qualitative flow features for the Newtonian fluids essentially remain unaltered for these scaled geometric configurations. However, flow of shear-rate dependent fluids varies for these scaled geometries because flow characteristics are a function of not only the Reynolds number but also the local shear-rate field that in turn is a function of the geometric length scale of the problem. Considerable variation in the computed wall shear-stress is observed between the Newtonian fluids and shear-rate dependent fluids, especially up to the shear-rate range of 400 s⁻¹. Time dependent features of the method are highlighted via the vortex-shedding problem. Stability and accuracy of the numerical results are presented on relatively cruder meshes at substantially reduced computational costs.

Acknowledgments

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References