A stabilized mixed finite element method for Darcy flow

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Abstract

We develop new stabilized mixed finite element methods for Darcy flow. Stability and an a priori error estimate in the “stability norm” are established. A wide variety of convergent finite elements present themselves, unlike the classical Galerkin formulation which requires highly specialized elements. An interesting feature of the formulation is that there are no mesh-dependent parameters. Numerical tests confirm the theoretical results.

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1. Introduction

We consider the problem of Darcy flow which is of considerable practical importance in civil, geotechnical, and petroleum engineering. Currently, numerical methods are based on two different approaches: One involves a primal, single-field formulation for pressure; and the other employs a mixed two-field formulation in which pressure and velocity are variables.

The primal formulation amounts to solving a Poisson problem for pressure with a rough coefficient (i.e., the ratio of permeability to viscosity). This can be done with adequate accuracy using existing finite element methodology. However, it is the derived flux, that is, the velocity, which is of primary interest. In this formulation it is obtained by taking the gradient of pressure and multiplying it by the rough coefficient. There is typically a loss of accuracy in the process and, additionally, mass conservation is not guaranteed. Consequently, this basic approach has not proved adequate for practical engineering applications. An improved variant on this theme, which involves a post-processing of the velocity field to improve accuracy and enforce mass conservation, has proved to be a viable option (see, e.g., [41,42,44] and references therein).

The more popular approach in applications so far has been based on the mixed formulation. (In petroleum engineering, advective–diffusive equations for concentrations are coupled to the Darcy flow equation. For examples of the use of mixed finite elements see [2,14,15,17–19,51,52] and references therein.) The classical mixed variational formulation is posed in terms of the function spaces $L^2(\Omega)/\mathbb{R}$ and $H(\text{div}, \Omega)$
for the pressure and velocity, respectively. \( L^2(\Omega)/\mathbb{R} \) is the space of Lebesgue square-integrable functions defined on the domain \( \Omega \), modulo a constant. \( H(\text{div}, \Omega) \) is the space of Lebesgue square-integrable vector fields whose divergence is also Lebesgue square-integrable. See Brezzi and Fortin [8] for elaboration. It has been a challenge to develop finite element approximations of these spaces, which satisfy the celebrated Babuska–Brezzi, or inf–sup, stability condition [1,3]. Nevertheless, elegant solutions to this problem have been attained [4–7,45,46,48,50]. For a definitive summary of these developments, the reader is referred to the standard reference on mixed methods: Brezzi and Fortin [8]. These discrete spaces have been used successfully in numerous applications. Good accuracy has been attained for both velocity and pressure, and mass conservation is achieved locally (i.e., element-wise) as well as globally. However, this approach also has its drawback: complexity. Different interpolations are required for pressure and velocity (and concentration in simulation of miscible displacements in porous media) and implementation is particularly complicated in three dimensions. Velocities are required to have continuous normal components across element interfaces, whereas tangential components are discontinuous. Pressure fields are discontinuous and must not be of too high order, otherwise the inf–sup condition is violated. Only normal velocity degrees of freedom are present on element interfaces while all velocity degrees of freedom are present in element interiors. Within the classical mixed variational framework, this is the price one pays for success.

In this work we reconsider the mixed formulation of Darcy flow. In our opinion, it would represent a significant simplification of the problem if standard, continuous and discontinuous finite element spaces could be made to work [27]. Additionally, the approximations of \( H(\text{div}, \Omega) \) are not applicable to certain generalizations of Darcy flow, such as the Brinkman model of lubrication theory in which a Laplacian of velocity is also present.

A framework for pursuing our objectives is the theory of stabilized methods. (The literature has become enormous. For a sample of early works, see [10,12,20–22,24,25,28–32,35–37,39,40]. Texts treating stabilized methods are Quarteroni and Valli [47] and Roos et al. [49]). The philosophy of stabilized methods is to strengthen classical variational formulations so that discrete approximations, which would otherwise be unstable, become stable and convergent. Success has been achieved on a wide variety of problems, and this is the approach we have adopted herein. We introduce two new stabilized variational formulations for Darcy flow. The first accommodates continuous velocity and pressure interpolations; the second accommodates continuous velocities and discontinuous pressure.

The natural “stability norm” provides \( L^2(\Omega) \) control for the velocity in both cases. In the continuous pressure case, the pressure is stable in \( H^1(\Omega)/\mathbb{R} \). Thus we can prove convergence for all elements of this type: Any combination of continuous velocity and pressure converges in the stability norm.

In the discontinuous pressure case, the stability norm provides \( \tilde{H}^1(\Omega)/\mathbb{R} \) control of pressure, where \( \Omega \) is the union of element interiors. This means that the pressure is stable up to the constant part on each element. We need to rely on satisfaction of the inf–sup condition to control the constants. Fortunately, this is a well-studied problem and it is known that if the velocity space is “sufficiently large” stability is attained. Precise requirements for standard element families are given by Franca and Stenberg [25] (see also [22]).

The discontinuous pressure elements satisfy mass conservation locally and globally. The continuous pressure elements satisfy mass conservation globally, and locally in a sense described in Hughes et al. [33].

An interesting aspect of the new variational formulations is that no mesh-dependent parameters are present. This may be contrasted with most stabilized methods in which mesh-dependent parameters appear which may be thought of as arising from elimination of unresolved scales in a multiscale decomposition of the solution (see [26,34] for elaboration). The only other example of this type we are aware of is presented in Hughes and Brezzi [28]. In this respect, the new formulations are very clean and may be useful in generating numerical methods other than finite element methods, such as, for example, ones based on pseudo-spectral approximations (see, e.g., [13]).
We perform fairly extensive numerical tests involving two-dimensional equal-order velocity–pressure elements. Linear and quadratic triangles and quadrilaterals are tested. Both continuous and discontinuous pressure elements are evaluated. Convergence tests involving a smooth solution yield results which confirm the theoretical error estimate for all elements considered. This estimate involves the stability norm and for the cases considered it does not say whether or not optimality is achieved in \( L_2(\Omega) \). Our numerical results indicate that optimality is attained for several cases. It turns out that \( L_2(\Omega) \) optimality for velocities can be guaranteed from the stability norm error estimate by selecting the pressure interpolation to be one polynomial order higher than that used for the velocities. However, we did not attempt to verify this result numerically.

We also performed several tests of robustness involving elliptic singularities (the five-spot problem) and discontinuous coefficients associated with a checkerboard domain. Distorted and unstructured meshes were also tested. All elements governed by the theory are shown to be very robust.

We also numerically studied the four-node quadrilateral element with discontinuous pressure which lies outside our theory. The velocity space is not sufficiently rich to control the constant part of the pressure in each element. The behavior of the pressure for this element can be understood by considering the constant pressure/bilinear velocity quadrilateral within the classical Galerkin formulation, which has been thoroughly analyzed. It is known that certain pressure instabilities can be aroused under certain circumstances. The checkerboard instability is legendary in the engineering literature \([27,38,43]\). Nevertheless, it remains one of the most popular and widely used elements in engineering for a variety of practical reasons. Our formulation of the equal-order bilinear quadrilateral guarantees stability of the element pressures modulo the constant part on each element, so it is reasonable to anticipate the same type of problems as one encounters for the constant pressure/bilinear velocity quadrilateral. Our convergence tests involving a smooth solution reveal it behaves optimally in velocity and pressure. On the other hand, in one of our robustness tests some pressure oscillations are apparent.

An outline of the remainder of the paper follows: In Section 2 we describe the formulation in which the velocities and pressures will be approximated by continuous interpolations. (Strictly speaking, we only require continuity of the normal component of velocity across element interfaces, but we do not pursue this idea herein.) We begin with a review of the classical variational formulation of Darcy flow and follow with our stabilized formulation in which the pressure is assumed to be in \( H^1(\Omega)/\mathbb{R} \). We then describe the finite element formulation and establish the fundamental error estimate. In Section 3 we present the formulation in which pressure is only assumed to be in \( H^1(\Omega)/\mathbb{R} \). This formulation accommodates discontinuous pressures. In Section 4 we present results from the numerical tests, and in Section 5 we draw conclusions.

2. Mixed formulation with continuous velocity and pressure

Let \( \Omega \subset \mathbb{R}^{n_{sd}} \) be an open bounded region with piecewise smooth boundary \( \Gamma \). The number of space dimensions, \( n_{sd} \), is equal to 2 or 3. Darcy’s law for the flow of a viscous fluid in a permeable medium, and conservation of mass, are written as follows:

\[
\mathbf{v} = -\frac{\kappa}{\mu} \left( \nabla p + \frac{\rho}{g_c} \mathbf{g} \right) \quad \text{on } \Omega, \tag{1}
\]

\[
\text{div} \mathbf{v} = \varphi \quad \text{on } \Omega, \tag{2}
\]

\[
\mathbf{v} \cdot \mathbf{n} = \psi \quad \text{on } \Gamma, \tag{3}
\]

where \( \mathbf{v} \) is the Darcy velocity vector, \( p \) is the pressure, \( \mathbf{g} \) is the gravity vector, \( \varphi \) is the volumetric flow rate source or sink, \( \psi \) is the normal component of the velocity field on the boundary, \( \mu > 0 \) is the viscosity, \( \kappa > 0 \)
is the permeability, \( \rho > 0 \) is the density, \( g_c \) is a conversion constant, and \( n \) is the unit outward normal vector to \( \Gamma \). It is apparent from (2) and (3) that the prescribed data \( \varphi \) and \( \psi \) must satisfy the constraint \( \int_{\varOmega} \varphi \, d\Omega = \int_{\Gamma} \psi \, d\Gamma \).

2.1. The classical weak formulation

Let

\[
\varmathcal{V} = H(\text{div}, \varOmega) \vDash \{ \mathbf{v} \, | \, \mathbf{v} \in (L_2(\varOmega))^{\text{nd}}, \text{div} \mathbf{v} \in L_2(\varOmega), \triangle \mathbf{v} \cdot n = \psi \in H^{1/2}(\Gamma) \},
\]

\[
\varmathcal{P} = L_2(\varOmega) \setminus \mathbb{R} \vDash \{ p \, | \, p \in L_2(\varOmega), \int_{\varOmega} p \, d\Omega = 0 \}.
\]

For further elaboration on these spaces, see Brezzi–Fortin [8].

We assume \( \kappa, \mu, \rho, g_c, g, \varphi \) and \( \psi \) are given. The classical weak formulation of (1)–(3) is: Find \( \mathbf{v} \in \varmathcal{V} \), \( p \in \varmathcal{P} \), such that, for all \( \mathbf{w} \in \varmathcal{V} \), \( q \in \varmathcal{P} \),

\[
\left( \mathbf{w}, \frac{\mu}{\kappa} \mathbf{v} \right) - (\text{div} \mathbf{w}, p) + (q, \text{div} \mathbf{v}) = -\left( \mathbf{w}, \frac{\rho}{g_c} g \right) + (q, \varphi),
\]

where \((\cdot, \cdot)\) is the \( L_2(\varOmega) \) inner product. For sufficiently regular data, the weak formulation is known to possess a unique solution.

For future reference, it is convenient to rewrite (6) as follows: Let \( \mathscr{Y} = \varmathcal{V} \times \varmathcal{P} \), \( \mathcal{V} = \{ \mathbf{v}, p \} \) and \( \mathcal{W} = \{ \mathbf{w}, q \} \). Find \( \mathbf{V} \in \mathscr{Y} \), such that, for all \( \mathbf{W} \in \mathscr{Y} \),

\[
B(\mathbf{W}, \mathbf{V}) = L(\mathbf{W}),
\]

where

\[
B(\mathbf{W}, \mathbf{V}) = \left( \mathbf{w}, \frac{\mu}{\kappa} \mathbf{v} \right) - (\text{div} \mathbf{w}, p) + (q, \text{div} \mathbf{v}),
\]

\[
L(\mathbf{W}) = -\left( \mathbf{w}, \frac{\rho}{g_c} g \right) + (q, \varphi).
\]

Remark: This formulation has served as the basis of the Galerkin finite element method. It is known that only certain combinations of velocity and pressure interpolations are stable. In the sequel we formulate a weak formulation which is inherently more stable, and accommodates a greater variety of stable interpolations, many of which are known to be unstable in the classical formulation.

2.2. A stabilized weak formulation

Let

\[
\varmathcal{Q} = H^1(\varOmega) \setminus \mathbb{R} \vDash \{ q \, | \, q \in H^1(\varOmega), \int_{\varOmega} q \, d\Omega = 0 \},
\]

\[
\mathcal{Z} = \varmathcal{V} \times \varmathcal{Q}.
\]

The stabilized weak formulation is: Find \( \mathbf{V} \in \mathcal{Z} \), such that, for all \( \mathbf{W} \in \mathcal{Z} \),

\[
B_{\text{stab}}(\mathbf{W}, \mathbf{V}) = L_{\text{stab}}(\mathbf{W}),
\]
where
\[ B_{\text{stab}}(W, V) = B(W, V) + \frac{1}{2} \left( -\frac{\mu}{\kappa} w + \nabla q \right) \cdot \frac{\kappa}{\mu} \left( \frac{\mu}{\kappa} v + \nabla p \right) \],

\[ L_{\text{stab}}(W) = L(W) - \frac{1}{2} \left( -\frac{\mu}{\kappa} w + \nabla q \right) \cdot \frac{\kappa}{\mu} \left( \frac{\rho}{\kappa} g \right) \],

and \( B(W, V) \) and \( L(W) \) are given by (8) and (9), respectively.

### 2.3. The finite element method

Consider a decomposition of \( \Omega \) into open non-overlapping element subdomains, \( \Omega^e, e = 1, 2, \ldots, n_{el} \), where \( n_{el} \) is the total number of elements, satisfying the minimum angle condition, such that
\[
\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e. \tag{15}
\]

Let
\[
\mathcal{X} = \{ v^h | v^h \in \mathcal{X}, v^h \in (C^0(\Omega))^n_{ad}, v^h|_{\partial \Omega} \in (C^0(\Omega^e))^n_{ad}, e = 1, 2, \ldots, n_{el} \}, \tag{16}
\]
\[
\mathcal{Q} = \{ q^h | q^h \in \mathcal{Q}, q^h \in C^0(\Omega), q^h|_{\partial \Omega} \in C^0(\Omega^e), e = 1, 2, \ldots, n_{el} \}, \tag{17}
\]
\[
\mathcal{X}^h = \mathcal{X} \times \mathcal{Q}. \tag{18}
\]

Note that \( \mathcal{X}^h \) is a closed linear subspace of \( \mathcal{X} \).

The discrete approximation is then stated as follows: Find \( v^h = \{ v^h, p^h \} \in \mathcal{X}^h \), such that, for all \( W^h = \{ w^h, q^h \} \in \mathcal{X}^h \),
\[
B_{\text{stab}}(W^h, v^h) = L_{\text{stab}}(W^h). \tag{19}
\]

We now examine the convergence properties of this method.

### 2.4. Error analysis

Let \( \| \cdot \| \) denote the \( L_2(\Omega) \) norm. We define the “stability norm” by
\[
\| W \|_{\text{stab}} = \left\{ \frac{1}{2} \left( \left\| \left( \frac{\mu}{\kappa} \right)^{1/2} w \right\|^2 + \left\| \left( \frac{\kappa}{\mu} \right)^{1/2} \nabla q \right\|^2 \right) \right\}^{1/2} \forall W \in \mathcal{X}. \tag{20}
\]

**Lemma 1** (Stability). For all \( W \in \mathcal{X} \),
\[
B_{\text{stab}}(W, W) = \| W \|_{\text{stab}}^2. \tag{21}
\]

**Proof.** The proof follows immediately from the definition of \( B_{\text{stab}}(\cdot, \cdot) \). \( \square \)

**Lemma 2** (Consistency). For all \( W^h \in \mathcal{X}^h \),
\[
B_{\text{stab}}(W^h, E) = 0, \tag{22}
\]

where \( E = V^h - V \) is the error in the finite element solution.

**Proof.** Substituting \( W^h \in \mathcal{X}^h \) for \( W \) in (12) and subtracting from (19) yields the result. \( \square \)
2.5. Interpolation estimate

Let \( \mathbf{V}^h \in \mathcal{Z}^h \) be an interpolate of \( \mathbf{V} \in \mathcal{Z} \). We decompose the error in the standard way:

\[
\mathbf{E} = \mathbf{V}^h - \mathbf{V} = (\mathbf{V}^h - \mathbf{V}) + (\mathbf{V} - \mathbf{V}) \tag{23}
\]

where \( \mathbf{E}^h = \{e^h_r, e^h_p\} \in \mathcal{Z}^h \) and \( \mathbf{H} = \{\eta_r, \eta_p\} \in \mathcal{Z} \). We have the following standard estimate for the interpolation error \( \mathbf{H} \):

\[
\|\mathbf{H}\|_{\text{stab}} \leq \tilde{C}_1 \left( \frac{h}{L} \right)^{k+1} |v|_{k+1} + \tilde{C}_2 \left( \frac{h}{L} \right)^l |p|_{l+1}, \tag{25}
\]

where \( h \) is the mesh parameter, \( L \) is a characteristic dimension of \( \Omega \), \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are constants independent of \( h, v \) and \( p \), but depend on \( \mu \) and \( \kappa \) in the following way:

\[
\tilde{C}_1 = C_1 \sup_{x \in \Omega} \left( \frac{\mu(x)}{\kappa(x)} \right)^{1/2}, \tag{26}
\]

\[
\tilde{C}_2 = C_2 \sup_{x \in \Omega} \left( \frac{\kappa(x)}{\mu(x)} \right)^{1/2} L^{-1}, \tag{27}
\]

in which \( C_1 \) and \( C_2 \) are non-dimensional constants, and \( | \cdot |_r \) is the \( r \)th Sobolev seminorm, defined by

\[
| \cdot |_r^2 = | \cdot |_{H^r(\Omega, L)}^2 = \int_{\Omega} \left( L^s \nabla \cdots \nabla ( \cdot ) \right)^2 \text{d}\Omega. \tag{28}
\]

Theorem.

\[
\|\mathbf{E}\|_{\text{stab}} \leq C \|\mathbf{H}\|_{\text{stab}}, \tag{29}
\]

where \( C \) is a non-dimensional constant.

Corollary.

\[
\|\mathbf{E}\|_{\text{stab}} \leq C \left( \frac{h}{L} \right)^{k+1} |v|_{k+1} + \tilde{C}_2 \left( \frac{h}{L} \right)^l |p|_{l+1}. \tag{30}
\]

Proof of Corollary. The result follows from (24) and (25). \( \square \)

Proof of Theorem.

\[
\|\mathbf{E}\|_{\text{stab}}^2 = B_{\text{stab}}(\mathbf{E}, \mathbf{E}) \quad \text{(stability)},
\]

\[
= B_{\text{stab}}(\mathbf{E}^h + \mathbf{H}, \mathbf{E}) \quad \text{(definition of \( \mathbf{E} \))},
\]

\[
= B_{\text{stab}}(\mathbf{E}^h, \mathbf{E}) + B_{\text{stab}}(\mathbf{H}, \mathbf{E}) \quad \text{(bilinearity)},
\]

\[
= B_{\text{stab}}(\mathbf{H}, \mathbf{E}) \quad \text{(consistency)}. \tag{31}
\]
We now proceed to estimate $B_{\text{stab}}(H, E)$:

$$B_{\text{stab}}(H, E) = B_{\text{stab}}(\eta_r, \eta_p, e_r, e_p) = \left( \eta_r, \frac{\mu}{K} e_r \right) - \left( \text{div} \eta_r, e_p \right) + \left( \eta_p, \text{div} e_r \right)$$

$$- \frac{1}{2} \left( \eta_r, \frac{\mu}{K} e_r \right) - \frac{1}{2} \left( \eta_r, \nabla e_p \right) + \frac{1}{2} \left( \nabla \eta_p, e_r \right) + \frac{1}{2} \left( \nabla \eta_p, \frac{\mu}{K} \nabla e_p \right)$$

$$\leq \frac{1}{2} \left\{ \varepsilon_1 \left\| \left( \frac{\mu}{K} \right)^{1/2} \eta_r \right\|^2 + \frac{1}{\varepsilon_1} \left\| \left( \frac{\mu}{K} \right)^{1/2} e_r \right\|^2 + \varepsilon_2 \left\| \left( \frac{\mu}{K} \right)^{1/2} \eta_r \right\|^2 + \frac{1}{\varepsilon_2} \left\| \left( \frac{\mu}{K} \right)^{1/2} \nabla e_p \right\|^2 \right.$$  

$$+ \frac{1}{2} \varepsilon_3 \left\| \left( \frac{\mu}{K} \right)^{1/2} \nabla \eta_p \right\|^2 + \frac{1}{2} \left( \frac{\mu}{K} \right)^{1/2} e_r \left\| \nabla \eta_p \right\|^2 + \frac{1}{2} \left( \frac{\mu}{K} \right)^{1/2} \nabla e_p \right\|^2$$

$$+ \frac{1}{2} \varepsilon_5 \left\| \left( \frac{\mu}{K} \right)^{1/2} \eta_r \right\|^2 + \frac{1}{2} \left( \frac{\mu}{K} \right)^{1/2} \eta_r \left\| \nabla \eta_p \right\|^2 + \frac{1}{2} \left( \frac{\mu}{K} \right)^{1/2} \nabla e_p \right\|^2$$

$$+ \frac{1}{2} \varepsilon_7 \left( \frac{\mu}{K} \right)^{1/2} \nabla \eta_p \right\|^2 + \frac{1}{2} \left( \frac{\mu}{K} \right)^{1/2} \nabla e_p \right\|^2 \right\}.$$  

(32)

We select

$$2\varepsilon_1 = 2\varepsilon_3 = \varepsilon_4 = \varepsilon_6 = 10, \quad 2\varepsilon_2 = \varepsilon_5 = \varepsilon_7 = 6.$$  

(33)

With these, (32) becomes

$$B_{\text{stab}}(H, E) \leq \frac{1}{2} \left\{ \left\| E \right\|_{\text{stab}}^2 + 16 \left\| \left( \frac{\mu}{K} \right)^{1/2} \eta_r \right\|^2 + 13 \left\| \left( \frac{\mu}{K} \right)^{1/2} \nabla \eta_p \right\|^2 \right\} \leq \frac{1}{2} \left\{ \left\| E \right\|_{\text{stab}}^2 + 32 \left\| H \right\|_{\text{stab}}^2 \right\}.$$  

(34)

from which it follows that

$$\left\| E \right\|_{\text{stab}}^2 \leq 32 \left\| H \right\|_{\text{stab}}^2$$  

(35)

and so

$$\left\| E \right\|_{\text{stab}} \leq 4\sqrt{2} \left\| H \right\|_{\text{stab}}.$$  

(36)

Remarks

1. Integration-by-parts was used to obtain (32).
2. The error estimate (36) is seen to be quasi-optimal in the stability norm with constant $C \leq 4\sqrt{2}$.
3. This result means that any combination of continuous velocity and pressure interpolations converges.
   The convergence rate of velocity in the $L_2(\Omega)$—norm and the gradient of pressure in the $L_2(\Omega)$—norm is $\min\{k + 1, l\}$. It is interesting to note that to get convergence rate $l$ in the stability norm, $k$ need only be $l - 1$, that is, one order lower than for the pressure interpolation.
4. It is possible to add a term to control the divergence error. Consider

$$B_{\text{stab}}^{\text{div}}(W, V) = L_{\text{stab}}(H, V),$$  

where

$$B_{\text{stab}}^{\text{div}}(W, V) = B_{\text{stab}}(W, V) + \frac{\alpha}{2} \left( \text{div} \frac{\mu}{K} h^2 \text{div} V \right),$$  

(37)
\[ L_{\text{stab}}^{\text{div}}(W) = L_{\text{stab}}(W) + \frac{\alpha}{2} \left( \text{div} w, \frac{\mu}{\kappa} h^2 \varphi \right), \]  

where \( \alpha = O(1) \). The stability norm is \( h \)-dependent in this case, viz.,

\[ \| W \|^{\text{div}}_{\text{stab}} = \left\{ \| W \|_{\text{stab}}^2 + \frac{\alpha}{2} \left\| \frac{\mu}{\kappa} h \, \text{div} w \right\|^2 \right\}^{1/2}. \]

Following the previous developments, it can be shown that the finite element method for (37) yields

\[ \| E \|^{\text{div}}_{\text{stab}} \leq C \| H^{\text{div}}_{\text{stab}} \| \leq C \left( \hat{C}_1 \left( \frac{h}{L} \right)^{k+1} \| v \|_{k+1} + \hat{C}_2 \left( \frac{h}{L} \right)^l \| p \|_{l+1} \right). \]

The method strengthens convergence of the velocity to something stronger than \( L_2(\Omega) \), but not as strong as \( H(\text{div}, \Omega) \) due to the \( h^2 \)-factor.

5. It would be a simple matter to generalize this formulation to the Brinkman model,

\[ \frac{\mu}{\kappa} v + \nabla p = \mu \Delta v, \]  

\[ \text{div} v = 0, \]

where \( \Delta \) is the vector Laplace operator. The generalization of the standard weak formulation (see e.g., [8,27]) would involve the addition of a stabilization term,

\[ \frac{1}{2} \sum_{c=1}^{n_1} \int_{\Omega} \left( - \frac{\mu}{\kappa} w + \nabla g + \mu \Delta w \right) \tau^e \left( \frac{\mu}{\kappa} v + \nabla p - \mu \Delta v \right) d\Omega, \]

where \( \tau^e \) is a stabilization parameter of the form

\[ \tau^e = \min_{x \in \Gamma^e} \left\{ \frac{\beta h^2}{\mu(x)} \frac{\kappa(x)}{\mu(x)} \right\} \]

and \( \beta \) is a non-dimensional constant which depends on the element type. For discussions of stabilized methods for the Stokes problem, see [22,35,36].

6. We note that the stabilization terms in (13) can be written as

\[ \frac{1}{2} \left( - \mathcal{L}^* W, \frac{\mu}{\kappa} \left( \mathcal{L} V + \frac{\rho}{g_c} g \right) \right), \]

where

\[ \mathcal{L} V = \frac{\mu}{\kappa} v + \nabla p, \]  

\[ \mathcal{L}^* W = \frac{\mu}{\kappa} w - \nabla q, \]

in which \( \mathcal{L}^* \) is the adjoint of \( \mathcal{L} \). Stabilizations of this type are associated with the concepts of the variational multiscale formulation and residual-free bubbles, see [9,11,23,26,34]. See also [16].

7. Our opinion is that stabilization methods, such as Galerkin/least-squares (GLS), in which the stabilization terms take the form

\[ \frac{1}{2} \left( \mathcal{L} W, \frac{\mu}{\kappa} \left( \mathcal{L} V + \frac{\rho}{g_c} g \right) \right) + \frac{\alpha}{2} \left( \text{div} w, \frac{\mu}{\kappa} h^2 (\text{div} v - \varphi) \right) \]
are not as effective for the current problem as the methods proposed herein, namely (12) and (37). The only essential difference is the sign on the \((\mu/\kappa)w\) term, but this proves crucial. There is a marked drop in accuracy when (49) is used, and stability requires \(z\) to be sufficiently large, which potentially engenders velocity “locking”. This may be contrasted with (12) for which convergence is established without the additional \(a\)-term (i.e., \(a = 0\)). Observations of the inferior behavior of GLS compared with formulations such as (12) in situations similar to the current problem were originally made by Franca and Russo [23].

3. Mixed formulation with continuous velocity and discontinuous pressure

Let \(\Omega\) denote the union of element interiors,

\[
\Omega = \bigcup_{e=1}^{n_{\text{el}}} \Omega^e,
\]

and let

\[
\hat{\Omega} = H^1(\Omega)/H^1_0(\Omega) = \left\{ q | q \in H^1(\Omega), e = 1, 2, ..., n_{\text{el}}, \int_{\Omega} q \, d\Omega = 0 \right\},
\]

\[
\hat{\mathcal{Y}} = \mathcal{Y} \times \hat{\Omega}.
\]

Note that functions in \(\hat{\Omega}\) may be discontinuous across element boundaries.

The stabilized weak formulation is: Find \(V \in \mathcal{Y}\), such that, for all \(W \in \mathcal{Y}\),

\[
\bar{B}_{\text{stab}}(W, V) = \bar{L}_{\text{stab}}(W),
\]

where

\[
\bar{B}_{\text{stab}}(W, V) = B(W, V) + \frac{1}{2} \left( \left( -\frac{\mu}{\kappa} w + \nabla q \right), \frac{\kappa}{\mu} \left( \frac{\mu}{\kappa} v + \nabla p \right) \right)_\Omega,
\]

\[
\bar{L}_{\text{stab}}(W) = L(W) - \frac{1}{2} \left( \left( -\frac{\mu}{\kappa} w + \nabla q \right), \frac{\kappa}{\mu} \left( \frac{\rho}{g_c} \right) \right)_\Omega,
\]

in which \((\cdot, \cdot)_\Omega\) is the \(L_2(\Omega)\)-inner product, viz.,

\[
(\cdot, \cdot)_\Omega = \sum_{e=1}^{n_{\text{el}}} (\cdot, \cdot)_{\Omega^e},
\]

and \(B(W, V)\) and \(L(W)\) are given by (8) and (9), respectively.

The finite element spaces are

\[
\hat{\Omega}^h = \{ q^h | q^h \in Q, q^h|_{\Omega^e} \in p'(\Omega^e), e = 1, 2, ..., n_{\text{el}} \},
\]

\[
\hat{\mathcal{Y}}^h = \mathcal{Y}^h \times \hat{\Omega}^h.
\]

Note that \(\hat{\mathcal{Y}}^h \subset \hat{\mathcal{Y}}\).

The discrete approximation is then stated as follows: Find \(\mathbf{V}^h = \{ w^h, p^h \} \in \mathcal{Y}^h\), such that, for all \(\mathbf{W}^h = \{ w^h, q^h \} \in \hat{\mathcal{Y}}^h\),

\[
\bar{B}_{\text{stab}}(\mathbf{W}^h, \mathbf{V}^h) = \bar{L}_{\text{stab}}(\mathbf{W}^h).
\]
This time the stability norm is defined by

$$
\|W\|_{stab} = \{B_{stab}(W, W)\}^{1/2} = \left\{ \frac{1}{2} \left( \left\| \left( \frac{\mu}{\kappa} \right)^{1/2} w \right\|^2 + \left\| \left( \frac{\kappa}{\mu} \right)^{1/2} \nabla q \right\|_{H^1}^2 \right) \right\}^{1/2} \forall W \in \mathcal{F}. \quad (60)
$$

**Remarks**

1. This method is convergent for *discontinuous* pressure interpolations as long as the velocity is quadratic or higher, i.e., $k \geq 2$. This follows from a result of Franca and Stenberg [25]; see also [22]. The same error estimate holds in this case as before, namely Eq. (30), in terms of the stability norm defined by (60).

2. Elements of this type possess local flux conservation properties, i.e.,

$$
\int_{\Gamma^e} \text{div} \, v^h \, d\Omega = \int_{\Gamma^e} v^h \cdot n \, d\Gamma,
$$

where $\Gamma^e$ is the boundary of $\Omega^e$, $e = 1, 2, \ldots, n_{el}$. This property is considered to be very important in the petroleum reservoir simulation community.

3. Linear velocity triangles and tetrahedra, for example, require the addition of pressure jump terms as in Hughes and Franca [35]. Elements of this type do not achieve (61).

4. Bilinear velocity quadrilaterals are on the borderline of stability. Numerical results confirm this assessment. This can be understood by studying the constant pressure/bilinear velocity element ([27], Chapter 4). The element has been successfully used in many applications despite its fragile pressure stability.

5. Discontinuous pressures suggest the opportunity for eliminating pressure degrees-of-freedom at the element level. However, the element matrix corresponding to the pressure degrees-of-freedom is a Dirichlet matrix without boundary conditions. It corresponds to the term

$$
\int_{\Omega^e} \nabla q^h \cdot \nabla p^h \, d\Omega, \quad e = 1, 2, \ldots, n_{el},
$$

in the variational equation. This matrix is singular and has a kernel of dimension 1, corresponding to the element-wise constant mode. A small perturbation term can be introduced to eliminate this singularity and facilitate elimination of the pressures at the element level. The simplest way to do this is to add a regularization term of the form

$$
\varepsilon \int_{\Omega^e} \frac{1}{h_e^2} q^h p^h \, d\Omega, \quad e = 1, 2, \ldots, n_{el},
$$

where $h_e$ is the length scale associated with element $\Omega^e$, and $\varepsilon$ is a small, non-dimensional parameter which depends on computer word length. For example, we used a value $10^{-8}$ when computing in double precision. The motivation for the structure of the term can be seen by comparing (62) and (63). We want the magnitude of (63) to be a very small fraction of (62), but about the same fraction at all levels of mesh refinement. We wish to point out that, with an appropriate global equation solving strategy, it is not necessary to add this term. However, it proves convenient in some circumstances such as the case described. In addition, it eliminates the need to remove the constant part of the global pressure, which constitutes the kernel of the global operator.

### 4. Numerical examples

Fig. 1 shows the elements employed in the numerical studies. In each case the velocity is continuous. Both continuous and discontinuous pressure versions are investigated. The following quadrature rules were used...
throughout: linear quadrilaterals, $2 \times 2$ Gauss quadrature; quadratic quadrilaterals, $3 \times 3$ Gauss quadrature; linear triangles, 4 point quadrature; quadratic triangles, 7 point quadrature ([27], Chapter 3).

4.1. Convergence study

The first numerical simulation is a study of convergence rates. The domain under consideration is a biunit square, and the exact pressure solution is given by

$$p = \sin \frac{2\pi x}{L} \sin \frac{2\pi y}{L}. \quad (64)$$

The velocity field is computed from Darcy’s law (1), in which $\rho g / g_\nu$ is taken to be zero; $\phi$ is calculated from (2) by taking the divergence of the velocity field, and $\psi$ is calculated from (3) by taking the normal component of the velocity. In specifying the boundary-value problem, $\phi$ is integrated over $\Omega$ while $\psi$ is prescribed nodally at the boundary.

Fig. 2a and b show representative meshes used in the convergence studies for the linear quadrilateral and triangular elements, while Fig. 3a and b show the representative meshes for quadratic elements. For linear quadrilateral elements, the meshes employed consisted of 100, 400, 1600, 6400, and 14400 elements. The linear triangular element meshes consisted of exactly twice as many elements. For quadratic quadrilateral elements, the meshes employed consisted of 25, 100, 400, 1600, and 3600 elements. Again, the quadratic triangular element meshes consisted of twice as many elements. The element mesh parameter, $h$, is taken to be the edge length of the elements for the quadrilaterals, and the short-edge length for triangles.

4.1.1. Continuous pressure elements

In the first case (Figs. 4–7) we consider the stabilized formulation given in (12)–(14). Figs. 4 and 5 show the $L_2$-norm of the velocity field, and $L_2$-norm and $H^1$-seminorm of the pressure field for the linear quadrilaterals and triangles. Theoretical rates predicted by the theory are achieved. In fact, optimal $L_2$-rates of convergence for velocity and pressure are also attained for these cases. Figs. 6 and 7 present the corresponding rates for the quadratic elements. We see optimal rates for the pressure field. The rate for the

Fig. 1. A family of 2-D linear and quadratic elements.
velocity field is less than optimal in the $L_2$-norm, however, the attained rate is in accordance with the theoretical predictions.

In the second set of convergence studies (Figs. 8–11), an additional stabilization term of the form given in (37)–(39),

$$\frac{\alpha}{2} \left( \text{div } w, \frac{\mu}{\kappa} h^2 \text{div } v \right)$$

is introduced. A value of $\alpha = 1$ is used in these simulations. Again, optimal rates are attained for the linear elements (Figs. 8 and 9). The $L_2$-norm of pressure and velocity is sub-optimal for the quadratic elements (Figs. 10 and 11), but still conform with the theoretical predictions. However, this represents some
Fig. 4. Convergence rates for continuous pressure 4-node quadrilaterals ($\kappa/\mu = 1$).

Fig. 5. Convergence rates for continuous pressure 3-node triangles ($\kappa/\mu = 1$).

Fig. 6. Convergence rates for continuous pressure 9-node quadrilaterals ($\kappa/\mu = 1$).
Fig. 7. Convergence rates for continuous pressure 6-node triangles ($\kappa/\mu = 1$).

Fig. 8. Convergence rates for continuous pressure 4-node quadrilaterals with divergence stabilization ($\kappa/\mu = 1, \alpha = 1$).

Fig. 9. Convergence rates for continuous pressure 3-node triangles with divergence stabilization ($\kappa/\mu = 1, \alpha = 1$).
degradation compared with the $\alpha = 0$ case in that $L_2$-pressure errors for that case converged at optimal rate. In Figs. 10 and 11 the $L_2$-velocity errors and $H^1$-pressure errors are almost identical and their plotting symbols overlap.

The third set of convergence studies (Figs. 12–15) investigates the effect of parameters $\kappa$ and $\mu$ on the rates of convergence. The stabilized formulation given in (12)–(14) is again employed, and a value of $\kappa/\mu = 0.01$ is used. Theoretically predicted convergence rates are attained that conform with the computed rates for $\kappa/\mu = 1$ case in Figs. 4–7.

4.1.2. Discontinuous pressure elements

The fourth test case investigates the discontinuous pressure elements. As mentioned previously, the method is convergent for discontinuous pressure interpolations as long as $k \geq 2$. The computed rates for the 9-node quadrilaterals and 6-node triangles for $\kappa/\mu = 1$ are presented in Figs. 16 and 17. Optimal rates are noted for the pressure. The labels in the figures “$H^1 p$” means the pressure error in the $H^1(\bar{\Omega})$ norm in the discontinuous pressure case. In Figs. 18 and 19 we investigate the effect of the divergence stabilization term, (65). Once again, theoretically predicted rates are attained.
Fig. 12. Convergence rates for continuous pressure 4-node quadrilaterals ($\kappa/\mu = 0.01$).

Fig. 13. Convergence rates for continuous pressure 3-node triangles ($\kappa/\mu = 0.01$).

Fig. 14. Convergence rates for continuous pressure 9-node quadrilaterals ($\kappa/\mu = 0.01$).
Fig. 15. Convergence rates for continuous pressure 6-node triangles ($\kappa/\mu = 0.01$).

Fig. 16. Convergence rates for discontinuous pressure 9-node quadrilaterals ($\kappa/\mu = 1$).

Fig. 17. Convergence rates for discontinuous pressure 6-node triangles ($\kappa/\mu = 1$).
Fig. 18. Convergence rates for discontinuous pressure 9-node quadrilaterals with divergence stabilization ($\kappa/\mu = 1, \alpha = 1$).

Fig. 19. Convergence rates for discontinuous pressure 6-node triangles with divergence stabilization ($\kappa/\mu = 1, \alpha = 1$).

Fig. 20. Convergence rates for discontinuous pressure 4-node quadrilaterals ($\kappa/\mu = 1$).
It is interesting to note that even though the theory does not predict convergence for the 4-node quadrilaterals with discontinuous pressure fields, we have obtained optimal convergence rates in all norms for this element as well. Figs. 20 and 21 show the computed convergence rates for the discontinuous-pressure linear quadrilaterals with $\kappa/\mu = 1$ and 0.01, respectively. However, as anticipated the element-wise constant part of the pressure field for discontinuous pressure linear triangles is not stabilized by this formulation. Numerical results (not shown) exhibited violent pressure oscillations.

The exact pressure and the absolute value of the velocity field for this problem are shown in Fig. 22a and b, respectively. Fig. 23a and b presents the corresponding results for a representative mesh of 4-node elements, while Fig. 24a and b shows the results with the additional stabilization of the divergence using (37)–(39). The velocity profile shows some diffusive effect degrading the accuracy of the velocity field slightly. At this juncture we see no benefit in adding the divergence stabilization term. Optimal $L_2$-rates of convergence for velocity are lost for the quadratic elements in addition to the deficiency just noted. Because of these observations we will not consider divergence stabilization in the remainder of this paper.

![Fig. 21. Convergence rates for discontinuous pressure 4-node quadrilaterals ($\kappa/\mu = 0.01$).](image)

![Fig. 22. (a) Elevation plot of the exact pressure field ($\kappa/\mu = 1$). (b) Elevation plot of the absolute value of the exact velocity field ($\kappa/\mu = 1$).](image)
4.2. The five-spot problem

This section presents numerical results for a quarter of the five-spot problem. The biunit square domain shown in Fig. 25 has prescribed velocity at the source and the sink. Due to symmetry of the problem, zero normal flow is prescribed along the boundaries. We assumed the divergence of the velocity field, \( \varphi \), was a Dirac delta function acting at source and sink, with strength +1/4 and −1/4, respectively. We calculated an equivalent distribution of normal velocity, \( \psi \), and drove the problem with \( \psi \), setting \( \varphi = 0 \). In the case of linear elements, we assumed a linear distribution of \( \psi \) along the external edges of the corner elements, which is zero at the nodes adjacent to the corner nodes. This uniquely determines the distribution of \( \psi \) on the edges (see Fig. 26a). In the case of quadratic elements, we assume a parabolic distribution along the external

Fig. 23. (a) Elevation plot of the pressure field for a mesh of 400 4-node quadrilateral elements (\( \kappa/\mu = 1 \)). (b) Elevation plot of the absolute value of the velocity field for a mesh of 400 4-node quadrilateral elements (\( \kappa/\mu = 1 \)).

Fig. 24. (a) Elevation plot of the pressure field for a mesh of 400 4-node quadrilateral elements (\( \kappa/\mu = 1, \alpha = 1 \)). (b) Elevation plot of the absolute value of the velocity field for a mesh of 400 4-node quadrilateral elements (\( \kappa/\mu = 1, \alpha = 1 \)).

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Fig. 25. Schematic diagram of a quarter of the five-spot problem.

Fig. 26. Five-spot problem. Distribution of $\psi$ along the corner elements at the production wells. The distribution of $\psi$ at the injection wells is the same with opposite direction.

Fig. 27. Five-spot problem. Pressure contours for continuous pressure 3-node triangles and 4-node quadrilaterals.
edges of the corner elements, which is zero, and has zero derivative at the element vertex nodes away from the corner. Again, this uniquely defines the distribution of $\psi$ along the edge (see Fig. 26b). We used parameter values $\mu = 1$ and $\kappa = 0.5$.

4.2.1. Uniform meshes

The lower left-hand corner represents the source, or injection well, while the upper right hand corner represents the sink, or production well. The domain is discretized by 800 3-node triangles and 400 4-node quadrilaterals, and the pressure contours are shown in Fig. 27a and b. Fig. 28a and b show meshes composed of 200 6-node triangles and 100 9-node quadrilaterals and contour plots of pressure fields. The circular nature of the exact solution about the source and sink are apparent in the numerical results.

Fig. 29a and b present elevation plots of the exact pressure and absolute value of the velocity. It is important to note that the exact solution possesses singular behavior in the pressure and the velocity at the source and the sink. Fig. 30a and b presents elevation plots of the pressure and absolute value of the
Fig. 30. (a) Five-spot problem. Elevation plot of the pressure field for the continuous pressure 400 4-node element mesh. (b) Five-spot problem. Elevation plot of the velocity field for the continuous pressure 400 4-node element mesh.

Fig. 31. Five-spot problem. Pressure distribution along the diagonal for continuous pressure 4-node element meshes.

Fig. 32. Five-spot problem. Pressure distribution along the diagonal for various continuous pressure element types.
velocity for the uniform mesh of 4-node quadrilaterals. Fig. 31 shows pressure distribution along the diagonal for 4-node element meshes and results are compared with the exact solution. Likewise, Fig. 32 presents the pressure distribution along the diagonal for all equal-order continuous pressure elements. In all cases, the singular behavior of the exact solution is captured, an indication of the robustness of the formulation.

4.2.2. Distorted structured meshes

We performed numerical simulations over structured, distorted, graded meshes composed of 3- and 6-node triangles and 4- and 9-node quadrilaterals. The graded meshes together with the superposed pressure fields are shown in Figs. 33a and b and 34a and b. The degree of distortion has been kept the same for all the elements.
4.2.3. Distorted unstructured mesh composed of triangles and quadrilaterals

The next numerical simulation is for an unstructured mesh composed of 200 4-node quadrilaterals and 400 3-node triangles (Fig. 35). This simulation is a test of the formulation to accommodate mixing element types, combined with distortion in the presence of a singular solution. Once again, the approach appears to be very robust.

4.3. The checkerboard domain

The last simulation tests the formulation for cases in which there are abrupt changes in the permeability parameter. We consider the five-spot problem described earlier, now zoned as shown in Fig. 36. Fig. 37 shows the pressure profile with $\kappa/\mu = 1$ in zones I and IV, and $\kappa/\mu = 0.01$ in zones II and III, while Fig. 38 shows the pressure profile with $\kappa/\mu = 1$ in zones I and IV, and $\kappa/\mu = 0.001$ in zones II and III. In both the cases there is no oscillation in the pressure field, attesting to the robustness of the method.
Fig. 37. Five-spot problem for the checkerboard domain. The mesh consists of 400 4-node continuous pressure elements. Zones I and IV: $\kappa/\mu = 1$. Zones II and III: $\kappa/\mu = 0.01$.

Fig. 38. Five-spot problem for the checkerboard domain. The mesh consists of 400 4-node continuous pressure elements. Zones I and IV: $\kappa/\mu = 1$. Zones II and III: $\kappa/\mu = 0.001$.

Fig. 39 shows discontinuous pressure contours for discontinuous pressure 9-node quadrilaterals. There are no discernible pressure oscillations in this case. To verify the local mass flux conservation properties, we have plotted the element integrals of divergence in Fig. 40. Note that the element divergence errors are $O(10^{-8})$ or smaller. This is consistent with the regularization operator, (63), used in the calculations in which $\varepsilon$ was set to $10^{-8}$. Earlier we reported optimal convergence rates for 4-node quadrilaterals with discontinuous pressures. The discontinuous pressure contours and element divergences are presented in Figs. 41 and 42, respectively. This time there appears to be some oscillations in the pressure near the center of the mesh, confirming the borderline stability of the pressure field for this element. The element integrals of divergence are again $O(10^{-8})$, in keeping with the size of the regularization operator, (63), employed.
5. Conclusions

We presented a new stabilized mixed finite element method for Darcy flow. The method was proved stable and an a priori error estimate was obtained in the “stability norm”. The results pertain to a much wider variety of finite elements than the classical Galerkin formulation, which requires highly specialized elements. An interesting feature of the stabilized method is that no mesh-dependent parameters are involved. In one embodiment, it is shown that all continuous pressure elements in combination with all continuous velocities, converge. In another embodiment, it is argued that if the velocity space is sufficiently “rich”, then the method is convergent for all discontinuous pressure elements. A great variety of new possibilities present themselves.

Fig. 39. Five-spot problem for the checkerboard domain. Pressure contours for discontinuous pressure 9-node elements. Zones I and IV: $\kappa/\mu = 1$. Zones II and III: $\kappa/\mu = 0.01$.

Fig. 40. Five-spot problem for the checkerboard domain. Integrals of element divergence for discontinuous pressure 9-node elements. Zones I and IV: $\kappa/\mu = 1$. Zones II and III: $\kappa/\mu = 0.01$. 
Numerical tests were performed for equal-order linear and quadratic triangular and quadrilateral elements, with continuous velocities and continuous or discontinuous pressures. Convergence tests confirmed the stability-norm error estimate in all cases. In several cases, optimal $L_2(\Omega)$ convergence rates were observed as well. Robustness tests involving elliptic singularities and discontinuous coefficients were also performed. Elements which were stable, according to theory, were shown to be very robust. Elements, which were not stable, according to theory, were numerically fragile, or failed altogether.

In future works, we hope to explore various extensions and generalizations of the methodology presented herein.
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