A Multiscale Finite Element Method for the Fokker Planck Equation

Arif Masud, M. ASCE

University of Illinois at Chicago, Chicago, Illinois, 60607
amasud@uic.edu

Lawrence A. Bergman, M. ASCE

University of Illinois, Urbana-Champaign, 61801
lbergman@uiuc.edu

Abstract

In a recent paper, the authors developed the framework for multi-scale finite element methods for the solution of the multi-dimensional Fokker-Planck equation in stochastic structural dynamics. The Fokker-Planck equation governs the evolution of the transition probability density function of the response of a broad class of dynamical systems driven by Gaussian noise, and completely describes the system response process. Solutions employing the finite element method have heretofore been limited to dynamical systems of low dimension. The multi-scale finite element method is applied in an effort to develop a formulation that can yield higher accuracy on cruder spatial discretizations, thus reducing the computational overhead associated with the large scale problems that arise in higher dimensions. In this paper, the method is briefly reviewed and then applied to a 3 dimensional model problem.

Introduction

Determination of the response of dynamical systems driven by random noise has been a subject of continuing interest for a century, beginning with Einstein’s early work on the characterization of Brownian motion (1905). An extensive body of literature addressing engineering applications and their solutions exists, and the reader may wish to examine one or more recent review articles (e.g., Schueller, 1997) for an overview. Here, though, we will limit ourselves to numerical solution of the Fokker-Planck equation, which governs the evolution of the transition probability density function of the response process, assuming a memoryless system. For a dynamical system of dimension 2n, the Fokker-Planck equation is an initial-boundary value problem in 2n spatial dimensions plus time. The equation is convective-diffusive and non-self-adjoint. In general, analytical solutions exist, at least in terms of quadratures, only for scalar dynamical systems. There has been some success with systems of dimension two, particularly in determining stationary, or long term, behavior (Caughey, 1971; Soize, (1988)). This absence of analytical results has motivated the development of a body of approximation theory, one part of which involves the numerical solution of the Fokker-Planck equation by the finite element method. Among those who have contributed to the development of this class of solutions are Langley (1985), Langtangen (1991), and Bergman and co-workers (1982, 1992, 1995). Application of the method has been limited to systems of low dimension (i.e., less than five spatial dimensions), because of the computational overhead (number of equations, bandwidth, etc.) associated with high dimensionality. Recently, though, the authors have extended the capability of the finite element method to facilitate the analysis of large scale memoryless dynamical systems subjected to Gaussian excitations, retaining a sufficient number of states to capture all of the important behavior. Here we briefly present an overview of the multi-scale finite element method to be employed and examine some results for a 3 D model application.
Discussion

The Fokker-Planck, or forward Kolmogorov, equation corresponding to an \( n \)-dimensional dynamical system subjected to Gaussian white noise excitation is given in direct form as

\[
\dot{f} = -\nabla \cdot (\mathbf{z} f) + \frac{1}{2} \Delta (\mathbf{H} f) \quad \text{in } \Omega
\]

(1)

where \( \nabla = \text{gradient operator in } 'n' \text{ dimensions} \), \( \nabla \cdot = \text{divergence operator in } 'n' \text{ dimensions} \), \( \Delta = \text{Laplace operator in } 'n' \text{ dimensions} \), and \( \Omega \subset \mathbb{R}^n \) is an open bounded region in \( 'n' \) dimensions with piecewise smooth boundary \( \Gamma \). It will be assumed hereafter that \( n \geq 2 \). Let \( w \in H^1(\Omega) \) be the weighting function for the probability density function. The inner product of the weighting function with (1) gives the weak form of the equation

\[
(w, \dot{f}) + (w, \nabla \cdot (\mathbf{z} f)) + \frac{1}{2} (\nabla w, \mathbf{H} \nabla f) = 0
\]

(2)

where \( (\cdot, \cdot) \) is the \( L_2 \) inner product. Let \( \mathcal{V} \subset H^1(\Omega) \cap C^0(\Omega) \) represent the space of piecewise continuous trial solutions and weighting functions. Substituting the discrete counterparts of \( f \) and \( w \) into (3) gives the standard Galerkin form of the problem. In general, the standard Galerkin form requires a stabilized method to yield a solution of this class of problems.

We seek to develop a stabilized formulation in higher dimensions that can be applied to the Fokker-Planck equation. One of the earliest of the stabilized formulations applied to the convective-diffusion equation was that of Heinrich et al. (1977), which employed a “correction” to the usual weighting function based upon the known analytical solution of the one-dimensional convective-diffusion equation. Here, though, we approach the problem in a different way, that finds its roots in Hughes’ Variational Multi-scale method. We know that all numerical methods are based on the notion of the existence of underlying computational grids (i.e., a discrete set of points on which the field variables are approximated). The distance between these points is the characteristic length scale of discretization and is represented by \( 'h' \). This length scale, which is strictly an artifact of the discretization process, plays a crucial role in the solution procedure. For a well-defined problem and with a good numerical procedure, one is able to resolve the features of the solution that are larger than, or at least of the order of, the discretization scale \( 'h' \). However, features of the solution which are finer than \( 'h' \) are not captured. Within the context of Galerkin finite element methods, the only way to capture these fine scale features is to refine the mesh; i.e., to reduce the artificial length scale \( 'h' \). It is important to realize that if this approach is adopted, one very quickly reaches the limit of any given computational platform. Thus, we consider here whether it is possible to capture all features of the solution, both coarse and fine, on coarse meshes. In other words we endeavor to develop a numerical scheme that shows high accuracy on crude discretizations in an effort to reduce the computational overhead associated with large scale problems that arise in higher dimensions.
Let us assume that the total solution \( f \) is decomposable into
\[
f = f^h + f^e
\]
where \( f^h \) is the portion of the solution that can be obtained with a good numerical scheme and \( f^e \) is the part of the solution that is lost because it is finer than the length scale ‘h’. In other words, \( f^e \) represents the error in the solution. In Masud & Bergman (2003), we show a multi-scale procedure wherein we build this term back into the formulation, automatically giving a stabilized form of the problem and an explicit expression for the stabilization parameter. Since the stabilization term is not “designed” specifically for the problem, the method is readily applicable to approximation in n-dimensional space and is ideally suited to the Fokker-Planck equation in higher dimensions, where generalization of existing stabilization methods is not straightforward. A short summary of the method follows.

From the outset we presume the existence of multiple scales in the problem. We consider the bounded domain \( \Omega \) discretized into \( n_{\text{anel}} \) non-overlapping regions \( \Omega^e \) (element domains in 'n' dimensions) with boundaries \( \Gamma^e, e = 1, 2, \ldots n_{\text{anel}} \) such that
\[
\Omega = \bigcup_{e=1}^{n_{\text{anel}}} \Omega^e
\]
and denote the union of element interiors and element boundaries by \( \Omega' \) and \( \Gamma' \), respectively. We assume an overlapping sum decomposition of the PD (probability density) field and weighting function into coarse or resolvable-scales and fine or subgrid-scales. From a physical viewpoint, the fine scales can be viewed as components associated with regions of high gradient in the PD field, and the vector \( x \) represents an ‘n’ dimensional space.
\[
f(x) = \underbrace{\tilde{f}(x)}_{\text{coarse}} + \underbrace{f'(x)}_{\text{fine}}, \quad w(x) = \underbrace{w'(x)}_{\text{coarse}} + \underbrace{w'(x)}_{\text{fine}}
\]
We further assume that the subgrid scales, although non-zero within the elements, vanish identically over the element boundaries, i.e., \( f' = w' = 0 \) on \( \Gamma' \).

Substitution of the overlapping-sum-decomposed trial solution and weighting function into the standard variational form (2) is the point of departure from the conventional Galerkin formulation
\[
\left( \nabla \tilde{w} + w', \tilde{f} + f' \right) + \left( \nabla w' + w', \nabla \cdot (\mathbf{z} (\tilde{f} + f')) \right) + \frac{1}{2} \left( \nabla (\tilde{w} + w'), \mathbf{H} \nabla (\tilde{f} + f') \right) = 0
\]
Assume that \( f' \) is represented by piecewise polynomials of sufficiently high order, continuous in \( x \) but discontinuous in time. In particular, \( f' \) is assumed to be composed of piecewise constant-in-time functions. Therefore \( \tilde{f} = \tilde{f}' \). Accordingly,
we are left with two problems, one in the coarse scale and the other in the fine scale, respectively,

\[(\bar{w}, \bar{f}) + (w, \nabla \cdot (z(\bar{f} + f'))) + \frac{1}{2} (\nabla \bar{w}, H \nabla (\bar{f} + f')) = 0 \quad (8)\]

\[(w', \acute{f}) + (w', \nabla \cdot (z(\acute{f} + f'))) + \frac{1}{2} (\nabla w', H \nabla (\acute{f} + f')) = 0 \quad (9)\]

The weighting function slot in \( W' \) contains only the fine scale components. The general idea at this point is to solve the fine scale problem to obtain the fine scale solution \( f' \) which can then be substituted into (8), thereby eliminating the fine scales yet retaining their effect.

Further details of the method including convergence studies, and application to the solution of the Fokker-Planck equation, may be found in Masud and Bergman (2003). The remainder of this paper will be devoted to results of a model problem: the 3 dimensional linear oscillator subjected to external white noise.

**Results**

A three state linear stochastic dynamical system serves as a benchmark problem to assess the accuracy and stability of the developed formulation.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = 
\begin{bmatrix}
x_2 \\
-\gamma \omega_0 x_1 - 2 \xi \omega_0 x_2 - \varepsilon x_1^3 + x_3 \\
-\alpha x_3
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} w(t) \quad (10)
\]

Where \( w(t) \) is a Gaussian white process with

\[
E[w(t)] = 0, \quad E[w(t_1)w(t_2)] = 2D \delta(t)
\]

for \( t = t_1 - t_2 \). For this case, an exact stationary solution of the Fokker-Planck equation exists and is given by

\[
f(x) = \left((2\pi)^{3/2} \Gamma\right)^{1/2} \exp\left(-\frac{1}{2} x^\top \Gamma^{-1} x\right)
\]

where \( \Gamma \) is the stationary covariance matrix of the system (10). Parameters employed are: \( \varepsilon = 1, \gamma = \pm 1, \omega_0 = 1, \xi = 0.2, \alpha = 1, D = 0.4 \). The computational domain is \(-4 < x_i < 4\). The mesh employed in the simulation is composed of 24x24x24 8-node brick elements. We have used the Crank-Nicholson algorithm with \( \Delta t = 0.01 \). The boundary conditions employed here are \( f=0 \) at all the boundaries.

Two problems are considered. In the first case the initial condition is a Gaussian distribution with standard deviation of unity and initial variance 0.2, centered at \((x,y,z) = (0,0,0)\). Figure 1a shows the evolution of the probability density function at \((0,0,0)\). In the second case the initial condition is a Gaussian distribution with standard deviation of unity and initial variance 0.2, centered at \((x,y,z) = (-1.5,-1.5,-1.5)\). Figure
1b shows the corresponding evolution of the probability density function at (0,0,0). In both the cases the converged solution attains the exact value of 0.241.

Conclusions

We have developed a multi-scale formulation for the Fokker-Planck equation of a class of oscillators in $R^n$. The key point of the method is a multiscale decomposition of the field variable into resolvable/coarse and unresolvable/fine scales. The proposed multiscale method leads to a stabilized formulation which is free of any user-defined parameters. Since the fine or the subgrid scales in the formulation are proportional to the grid scale residual, the method is consistent. We have tested the formulation on linear and quadratic triangular and quadrilateral elements and have obtained optimum numerical convergence rates in the norms considered. The formulation is applied it to a model problem, the three-dimensional linear oscillator, for which an exact solution exists. The computational result is a converged solution on a mesh of 24x24x24 8-node brick elements, representing an order of magnitude savings over previously reported converged results. Efforts continue toward further refinement in the procedure and eventual application to larger scale systems.

Figure 1. Probability density function for the linear oscillator at (0,0,0): (a) with IC at (0,0,0), (b) with IC at (-1.5,-1.5,-1.5).

References


Masud, A. and Bergman, L. A.