NUMERICAL ASSESSMENT OF SOME MEMBRANE ELEMENTS WITH DRILLING DEGREES OF FREEDOM†

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(Received 4 September 1993)

Abstract—A simple formulation of membrane finite elements with drilling degrees of freedom is presented. Elements based on this theory have been proved to converge and this is confirmed by numerical calculations. The incompatible modes approach is incorporated in the formulation to develop extremely accurate four-node elements with drilling degrees of freedom.

1. INTRODUCTION

In recent years there has been a revival of interest in elements possessing in-plane rotational degrees of freedom (also called drilling degrees of freedom, see Fig. 1). Membrane elements of this kind possess practical advantages in the analysis of shell structures and folded plates. For example, combining a plate bending element with a membrane element possessing drilling rotations forms a shell element in which each node has six degrees of freedom, three displacements and three rotations. Typical membrane finite elements do not possess the in-plane rotational degree of freedom and so, when combined with a plate element, form a shell element with only five degrees of freedom per node. Although it is possible to work in a locally defined five degree of freedom system at each node, numerous practical difficulties in programing and model construction must be overcome. Membrane finite elements with drilling degrees of freedom circumvent these problems. Thus, the presence of the sixth nodal degree of freedom is very appealing from an engineering point of view.

Numerous works have appeared in the engineering literature in the last few years in which successful approaches towards incorporating drilling rotations in membrane elements have been described [1–28]. It is interesting to note that most of the elements proposed involve a variety of special devices. The simplest and most commonly used remedy is the addition of a fictitious torsional-spring stiffness at each node. Although it is free to work in a locally defined five degree of freedom system at each node, numerous practical difficulties in programing and model construction must be overcome. Membrane finite elements with drilling degrees of freedom circumvent these problems. Thus, the presence of the sixth nodal degree of freedom is very appealing from an engineering point of view.

Numerous works have appeared in the engineering literature in the last few years in which successful approaches towards incorporating drilling rotations in membrane elements have been described [1–28]. It is interesting to note that most of the elements proposed involve a variety of special devices. The simplest and most commonly used remedy is the addition of a fictitious torsional-spring stiffness at each node. However, renders the numerical method inconsistent, possibly degrading its convergence properties. There have also been developments in the mathematics literature, where variational formulations that employ independent rotation fields have been studied [29–32]. Ideas of this kind go back to Reissner [33]. In [10] we endeavored to pursue this subject mathematically, with an aim of developing a theoretically sound and, at the same time, practically useful formulation for engineering applications.

In this paper we present a synopsis of the simplest theory we developed, along with some numerical results. We begin in Section 2 with a statement of the strong form of the boundary-value problem. Then we introduce a modified variational formulation which incorporates independent rotation fields. In Sections 4 and 5, we present a detailed mathematical analysis of the formulation, along with the major attributes of the theory. Higher accuracy on coarse meshes with lower-order elements is obtained by drilling elements with incompatible modes, developed in Section 6; this demonstrates the generality of the method proposed and supports the assertion that a standard element may be generalized to an element with drilling degrees of freedom by employing the methodology presented herein. Section 7 represents extensive numerical tests that verify the superior performance of these elements, and conclusions are drawn in Section 8.

2. STRONG FORM OF THE BOUNDARY-VALUE PROBLEM

Let $\Omega \subset \mathbb{R}^d$ be an open set with piecewise smooth boundary $\Gamma$; $d \geq 2$ denotes the number of spatial dimensions. We assume that $\Gamma$ admits the decomposition

$$\Gamma = \Gamma_e \cup \Gamma_b,$$

(2.1)

where

$$\Gamma_e \cap \Gamma_b = \emptyset;$$

(2.2)

† This research was supported by the U.S. Office of Naval Research under contract N00014-88-K-0446.
Fig. 1. Membrane element with drilling degrees of freedom.

$\Gamma_1$ and $\Gamma_2$ are the portions of the boundary with prescribed essential boundary conditions and prescribed tractions, respectively.

We wish to solve the equations of linear elasticity on $\Omega$, adopting a somewhat unconventional viewpoint, namely that the stress tensor $\sigma$ is not assumed to be symmetric. In addition to $\sigma$, our dependent variables are $u$, the displacement vector, and $\psi$, a skew-symmetric tensor representing the (“infinitesimal”) rotation, which complements the skew-symmetric part of $\sigma$ in our theory. We frequently employ the Euclidean decomposition of a second-rank tensor e.g.

$$\sigma = \text{symm} \sigma + \text{skew} \sigma$$  \hspace{1cm} (2.3)

where

$$\text{symm} \sigma = \frac{1}{2}(\sigma + \sigma^T)$$  \hspace{1cm} (2.4)

$$\text{skew} \sigma = \frac{1}{2}(\sigma - \sigma^T).$$  \hspace{1cm} (2.5)

We focus our attention on the Dirichlet problem. More complicated boundary conditions provide no essential difficulties and may be handled by standard means (see, e.g., [34]). The formal statement of the boundary-value problem under consideration is: Given $f$, the body force vector, find $u$, $\psi$, and $\sigma$ such that

$$\text{div } \sigma + f = 0$$  \hspace{1cm} (2.6)

$$\text{skew } \sigma = 0$$  \hspace{1cm} (2.7)

$$\psi = \text{skew } \text{grad } u$$  \hspace{1cm} (2.8)

$$\text{symm } \sigma = c \cdot \text{symm } \text{grad } u$$  \hspace{1cm} (2.9)

$$u = 0 \quad \text{on } \Gamma = \partial \Omega.$$  \hspace{1cm} (2.10)

Equations (2.6)-(2.10) are, respectively, the equilibrium equation, the symmetry condition for stress, the definition of the rotation in terms of the displacement gradient, the constitutive equation and the displacement boundary condition. The elastic moduli, $c = [c_{ijkl}]$, $1 \leq i, j, k, l \leq d$, are assumed to satisfy the following conditions:

$$c_{ijkl} = c_{klij}$$  \hspace{1cm} (2.11)

$$c_{ijkl}c_{jkl} > 0, \quad \forall c_{ijkl} 
eq 0.$$  \hspace{1cm} (2.12)

Summation of repeated indices is assumed to hold throughout. Equations (2.11) (2.13) are referred to as, respectively, the major symmetry, the minor symmetries, and positive-definiteness. In the isotropic case,

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$  \hspace{1cm} (2.14)

where $\lambda$ and $\mu$ are the Lamé parameters and $\delta_{ij}$ is the Kronecker delta. For the particular case of plane stress, $\lambda$ needs to be replaced by

$$\lambda = 2\mu(\lambda + 2\mu).$$  \hspace{1cm} (2.15)

In terms of Young’s modulus, $E$, and Poisson’s ratio, $\nu$, we recall that

$$2\mu = E/(1 + \nu)$$  \hspace{1cm} (2.16)

$$\lambda = \nu E/(1 - \nu^2).$$  \hspace{1cm} (2.17)

For simplicity, we assume the elastic moduli are constant.

3. DISPLACEMENT-TYPE MODIFIED VARIATIONAL FORMULATION OF LINEAR ELASTOSTATICS WITH INDEPENDENT ROTATION FIELDS

A number of variational formulations of the boundary-value problem (2.6)–(2.10) were studied in detail in [10]. An interesting fact which emerges from this study is that numerical methods based on the conventional functional [33] are unstable when convenient, equal-order interpolations are employed. These difficulties are circumvented by modifying the variational formulation. (Consult [10] for details and mathematical analyses.)

Spaces relevant to the boundary value problem are

$$V = \{v | v \in (H^1_0(\Omega)^d)\}$$  \hspace{1cm} (3.1)

$$W = \{\omega | \omega \in (L_2(\Omega))^d, \text{symm } \omega = 0\}$$  \hspace{1cm} (3.2)

where $L_2(\Omega)$ denotes the space of square-integrable functions on $\Omega$; $H^1(\Omega)$ denotes the space of functions in $L_2(\Omega)$ with generalized derivatives also in $L_2(\Omega)$; and $H^1_0(\Omega)$ is the subset of $H^1(\Omega)$ whose members satisfy zero boundary conditions. The simplest formulation derived in [10] is based upon a modified potential energy functional $\tilde{\Pi}: V \times W \to \mathbb{R}$, defined as

$$\tilde{\Pi}(v, \omega) = \frac{1}{2} \int_{\Omega} \text{symm grad } v \cdot c \cdot \text{symm grad } u \, d\Omega$$

$$+ \frac{1}{2} \int_{\Omega} |\text{skew grad } v - \omega|^2 \, d\Omega$$

$$- \int_{\Omega} v \cdot f \, d\Omega.$$  \hspace{1cm} (3.3)
The variational equation emanating from (3.3) is a symmetric bilinear form and
\[ 0 = D\Omega, (u, \psi) \cdot (v, \omega) \]
\[ = \int_\Omega (\text{symm grad } v) \cdot e \cdot (\text{symm grad } u) \, d\Omega \]
\[ + \int_\Omega (\text{skew grad } v - \omega)^T \cdot (\gamma (\text{skew grad } u - \psi)) \, d\Omega \]
\[ - \int_\Omega v \cdot f \, d\Omega \]
\[ - \int_\Omega v \cdot [\text{div}(e \cdot \text{symm grad } u)] \, d\Omega \]
\[ + \gamma (\text{skew grad } u - \psi) + f] \, d\Omega \]
\[ - \int_\Omega \sigma \cdot \omega^T \cdot (\gamma (\text{skew grad } u - \psi)) \, d\Omega. \] (3.4)

In this formulation, the quantity \( \gamma (\text{skew grad } u - \omega) \) is to be viewed as the skew-symmetric part of the stress tensor. The last term in (3.4) asserts that the skew-symmetric stresses are zero and the preceding term expresses equilibrium in terms of the symmetric and skew-symmetric stresses. In the corresponding discrete case, of course, the skew-symmetric stresses will not be in general identically zero and thus will play a role in the equilibrium conditions. The value of the parameter \( \gamma \) will be assigned subsequently.

### 4. MATHEMATICAL ANALYSIS OF THE VARIATIONAL PROBLEM

The variational form of problem (3.4) may be rewritten as follows:

**Problem \( D^h \):** Find \( \{u, \psi\} \in U \equiv V \times W \) such that
\[ B_r (u, \psi; v, \omega) = f\{v, \omega\}, \quad \forall\{v, \omega\} \in U, \] (4.1)
where
\[ B_r (u, \psi; v, \omega) = \int_\Omega (\text{symm grad } v) \cdot e \cdot (\text{symm grad } u) \, d\Omega \]
\[ + \int_\Omega (\text{skew grad } v - \omega)^T \cdot (\gamma (\text{skew grad } u - \psi)) \, d\Omega \]
\[ = \int_\Omega \sigma \cdot \omega^T \cdot (\gamma (\text{skew grad } u - \psi)) \, d\Omega. \] (4.2)

is a symmetric bilinear form and
\[ f\{v, \omega\} = \int_\Omega v \cdot f \, d\Omega \] (4.3)
is continuous.

Let \( V^h \) and \( W^h \) be finite-dimensional subspaces of \( V \) and \( W \), respectively. We wish to think of \( V^h \) and \( W^h \) as typical finite element spaces involving piecewise polynomial interpolations of degree \( k \geq 1 \) and \( l \geq 0 \), respectively, with a mesh parameter \( h \). The discrete problem corresponding to (4.1) is:

**Problem \( D^h \):** Find \( \{u^h, \psi^h\} \in U^h = V^h \times W^h \) such that
\[ B_r (u^h, \psi^h; v^h, \omega^h) = f\{v^h, \omega^h\}, \quad \forall\{v^h, \omega^h\} \in U^h. \] (4.4)
The following standard theorem (see, e.g., [35, 36]) characterizes the well-posedness of problems \( D \) and \( D^h \).

**Theorem.** Assume:

(i) \( B_r \) is continuous, i.e. there exists a constant \( \delta > 0 \) such that
\[ \|\omega - \omega^h\| \leq \delta \{u, \psi\} \|u, \psi\|_U, \quad \forall\{u, \psi\}, \{v, \omega\} \in U \] (4.5)
where
\[ \|v, \omega\|_U^2 = \|v\|_u^2 + \|\omega\|_w^2, \quad \forall\{v, \omega\} \in U \] (4.6)
\[ \|v\|_u^2 = \|\text{grad } v\|_u^2 = \int_\Omega |\text{grad } v|^2 \, d\Omega, \quad \forall v \in V \] (4.7)
\[ \|\omega\|_w^2 = \|\omega\|_w^2 = \int_\Omega |\omega|^2 \, d\Omega, \quad \forall \omega \in W; \] (4.8)

(ii) \( B_r \) is \( U \)-elliptic, i.e. there exists a constant, \( \eta > 0 \), such that
\[ B_r (u, \psi; v, \omega) \geq \eta \{v, \omega\} \|u, \psi\|_U^2, \quad \forall\{v, \omega\} \in U \] (4.9)

(iii) for each sufficiently regular \( \{u, \psi\} \), there exists \( \{\hat{u}^h, \hat{\psi}^h\} \in U^h \) such that
\[ \|u - \hat{u}^h\|_U \leq c(u)h^k \] (4.10)
\[ \|\psi - \hat{\psi}^h\|_w \leq c(\psi)h^{l+1}; \] (4.11)

where \( c(u) \) and \( c(\psi) \) are independent of \( h, k \) and \( l \).

Then there exist solutions \( \{u, \psi\} \) and \( \{u^h, \psi^h\} \) of problems \( D \) and \( D^h \), respectively, such that
\[ \|u - u^h\|_U \leq \frac{\delta}{\eta} \{v^h, \omega^h\} \|v^h, \omega^h\|, \quad \forall\{v^h, \omega^h\} \in U^h \] (4.12)
and
\[
\| u^r - u, \psi^r - \psi \|_{C^r} \leq c(u, \psi) h^r, \tag{4.13}
\]
where \( r = \min \{k, l + 1 \} \) and \( c(u, \psi) \) is independent of \( h \).

The verification of the continuity condition (4.5) is straightforward. The ellipticity condition (4.9) provides insight into choosing an appropriate value of \( \gamma \). Let
\[
2\varepsilon = \min_{\varepsilon > 0} \frac{\epsilon \cdot c \cdot \epsilon}{\| \epsilon \|_2^2}. \tag{4.14}
\]
In the isotropic case, \( \varepsilon \equiv \mu \). Let \( c_i \) denote the constant in Korn's inequality:
\[
\| \text{symm grad } v \|^2 \geq c_i \| \text{grad } v \|^2. \tag{4.15}
\]

For the case of \( v \in V \), it is a simple exercise to show that \( c_i = 1/2 \) (see, e.g., Gurtin[37], pp. 38-39). Clearly \( c_i \leq 1 \) in general. We proceed as follows:
\[
B; (v, \omega; v, \omega) \geq 2\varepsilon \| \text{symm grad } v \|^2 \tag{4.16}
\]
From (4.16), the choice \( \gamma < 2\varepsilon c_i \) seems appropriate. For \( \gamma = 2\varepsilon c_i \),
\[
B; (v, \omega; v, \omega) \geq 2\varepsilon c_i \| \text{symm grad } v \|^2 + \gamma \| \text{skew grad } v - \omega \|^2
\]
where
\[
\| \text{symm grad } v \|^2 + \| \text{skew grad } v - \omega \|^2
\]
\[
\geq 2\varepsilon c_i \| \text{symm grad } v \|^2 + \gamma \| \text{skew grad } v - \omega \|^2
\]
\[
+ \frac{\gamma}{2} \| \omega \|^2. \tag{4.17}
\]
In the last line we used the fact that \( c_i = 1/2 \) for the Dirichlet problem. By virtue of (4.17), we have verified (4.9) with \( \eta = \frac{\varepsilon}{2} \). It is interesting to note that in the formulation leading to the PEERS element [30] this same value of the parameter was chosen at the outset in defining the bilinear form. Note that \( \gamma \leq 2\varepsilon c_i = \xi \left( = \mu \right. \) in the isotropic case).

Equations (4.10) and (4.11) follow from standard interpolation theory [35], thereby completing the verification of (4.12) and (4.13).

5. SALIENT FEATURES OF THE FORMULATION

The theory of membrane elements with drilling degrees of freedom discussed in the preceding sections has a number of noteworthy attributes:

1. For equal-order interpolations, \( r = k \) in (4.13).
2. The solutions of problems \( D_r \) and \( D_r^h \) minimize \( \bar{n}_r \) over \( U \) and \( U^h \), respectively.
3. The solution of \( D_r^h \) is the best approximation in the energy norm, namely:
\[
\| v^h - u, \psi^h - \psi \| \leq \| v - u, \omega - \psi \|. \tag{5.1}
\]
where
\[
\forall \{ v^h, \omega^h \} \in U^h, \tag{5.2}
\]

4. The matrix problem will have the following structure:
\[
\begin{bmatrix}
K & K_{11} & K_{12} \\
K_{21} & K_{22} & 0
\end{bmatrix}
\begin{bmatrix}
d \\\n\theta
\end{bmatrix}
= \begin{bmatrix}
F \\\n0
\end{bmatrix} \tag{5.3}
\]
where \( d \) and \( \theta \) are the nodal displacements and rotations, respectively; \( F \) is the nodal force vector; \( K \) is the usual stiffness matrix due to the displacement interpolations; and \( K_{11}, K_{12} - K_{21}, \) and \( K_{22} \) are the stiffness contributions of the \( \gamma \)-term. The \( \theta \)s may be eliminated to form the reduced system
\[
(K + \Delta K)d = F, \tag{5.4}
\]
where
\[
\Delta K = K_{11} - K_{12} K_{22}^{-1} K_{21}. \tag{5.5}
\]
The matrix \( \Delta K \) is positive-semidefinite, thus a stiffening effect is introduced by the \( \gamma \)-term. For equal-order interpolations the effect is small. In particular, it is easy to see that for a single element \( \Delta K \equiv 0 \). Thus the so-called "individual element tests" will be passed by equal-order elements based on the present formulation, as long as the underlying pure-displacement model passes these tests. See, e.g., Bergan and Hanssen [38].

5. If the elastic coefficients vary from element to element, then \( \gamma \) should be selected locally.

6. The theory does not anticipate much sensitivity to the particular value of \( \gamma \) because the formulations are stable and convergent for all positive values of \( \gamma \). For the isotropic case, we use \( \gamma = \mu \), since this value balances the terms in the stability estimate and thus seems reasonable. Numerical results presented herein also support this choice.
6. TWO-DIMENSIONAL DRILLING ELEMENTS

The present work is aimed at exploiting a mathematically sound variational formulation to develop elements that are easy to implement. In this section we present two classes of drill elements emanating from the variational formulation (4.4).

6.1. Conforming elements

Elements that employ standard finite element interpolation functions are called conforming or compatible. In this class, the finite element spaces used in the discrete problem are proper subsets of the corresponding spaces used in the weak statement of the problem, and hence the primary unknowns are continuous. This fact is explicitly used to prove convergence for this class of elements.

Figure 2 shows a family of conforming elements with drilling degrees of freedom. The underlying displacement elements are the standard three-node and six-node triangles, T3 and T6, respectively, and the Lagrange quadrilaterals Q4 and Q9. The terminology T6-drill and Q9-drill is used as a general reference to the higher-order elements with rotation fields (equal- and unequal-order). Rotations may be interpolated one order lower than displacements without degrading the asymptotic rates of convergence, see (4.13), or engineering accuracy, but equal-order interpolations are convenient to implement and use. The elements in Fig. 2 are designated by their underlying displacement elements followed by D and the number of nodes in the rotation field. Thus, for example, T6D3 is a six-node triangle with quadratic displacements and linear rotations, and Q9D4 is a nine-node quadrilateral with biquadratic displacements and bilinear rotations.

6.2. Elements with incompatible modes

Numerical tests show that the response of standard four-node quadrilaterals employed in beam bending problems is overly stiff (see, e.g., [34], p. 243); this happens because a four-node element when subjected to a moment in the form of a couple responds in shear rather than in bending. If only one element is used through the thickness, the accuracy attained is extremely poor. The poor coarse-mesh accuracy of lower-order elements in modeling bending behavior motivates one to try to improve their behavior for engineering analysis.

In order to improve the response of the bilinear element in bending-dominated situations, Taylor et al. employed incompatible modes [26]. (For a detailed discussion of the subject see, e.g., [34], pp. 243–249.) In the incompatible modes technique, quadratic modes corresponding to the bending solution are added to the standard expansion in terms of biquadratic shape functions, as follows:

\[ u_i(x, y) = \sum_{a=4}^{s} N_a(\xi, \eta) u_a + \sum_{a=5}^{s} N_a(\xi, \eta) \alpha_a, \quad (6.1) \]

where the first four \( N_a \)s are the standard bilinear shape functions,

\[ N_i(\xi, \eta) = 1 - \xi^2 \quad (6.2) \]

\[ N_6(\xi, \eta) = 1 - \eta^2 \quad (6.3) \]

and the \( \alpha_a \)s are generalized displacements (as distinguished from the nodal displacements). There are no nodal points associated with the additional modes and the \( \alpha_a \)s may be thought of as internal element degrees of freedom. The incompatible modes result in the element displacements being discontinuous between nodes. Since the generalized displacements associated with the incompatible modes are internal to each element, they are eliminated at the element level via static condensation, leading to a 12 x 12 element stiffness matrix.

We developed drilling elements with incompatible modes (Fig. 3) to attain the coarse mesh accuracy of biquadratic elements in tests dominated by in-plane bending, without changing the basic four-node pattern of the bilinear element, employing the Taylor-Wilson version of incompatible modes (QM6). The first element (QM6D4-I) has incompatible modes only in the symmetric gradient of the displacement field while the skew-symmetric part of the displacement gradient and the rotation field are bilinear. The second element (QM6D4-2) has incompatible modes throughout the displacement field, while the rotation field is bilinear. The third element (QM6D6) has incompatible modes in both the displacement and the rotation fields. We will compare these elements to the Q4, Q4D4 and QM6 elements.

Fig. 2. Finite elements with drilling degrees of freedom.

Key: \( \bullet \) displacement node, \( O \) drilling rotation node.

Fig. 3. Drilling elements with incompatible modes.
Our first incompatible modes element, QM6D4-I, was developed along the lines of displacement elements, in which only symmetric gradients play a role. This treatment of the displacement gradient is inconsistent with our formulation of drill elements. To remedy this, QM6D4-2 was developed with incompatible modes throughout the displacement gradients. We also wanted to examine the effect of incorporating incompatible modes into the rotation field, resulting in a QM6D6 element. In the following numerical tests, the inconsistency inherent in QM6D4-1 gives rise to somewhat inferior results. The addition of incompatible modes in the rotations offers no advantage in the performance of DM6D6 over DM6D4-2 (while requiring a higher-order integration rule). Therefore, DM6D4-2 seems to be the best choice among these elements.

7. NUMERICAL SIMULATIONS

In this section we present numerical tests of our membrane elements. Unless otherwise specified, we employed a value of $\gamma = \mu$. In accordance with the theory, rotations are not constrained throughout the domain. In all the tests, plane stress conditions are assumed in force. These tests include various patch tests and beam bending problems which serve as standard benchmark problems in the solid mechanics literature [1-3, 6, 7, 9, 13, 17, 19, 21, 23-25, 28]. Also presented are numerical rate of convergence studies of these elements, which confirm the convergence rates predicted by the theory (4.13). The superior performance of these elements under severe geometrical distortions is also presented. An extensive numerical study shows the insensitivity of this formulation to appropriate values of the parameter $\gamma$.

Numerical integration rules used are sufficient to evaluate all terms exactly. (This requires using the $3 \times 3$ rule for the evaluation of the rotation term in DM6D6.) Stresses for all elements are evaluated at the integration point and then projected onto the nodes using a least-squares type smoothing procedure, which redefines the stress field in terms of the displacement shape functions (see, e.g., [34], pp.

![Fig. 4. Configurations for patch tests.](image-url)
Numerical assessment of membrane elements

Table I. Normalized displacements at point A in Fig. 4

<table>
<thead>
<tr>
<th>Element type</th>
<th>Uniform mesh</th>
<th>Skewed mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_1(A)$</td>
<td>$u_2(A)$</td>
</tr>
<tr>
<td>Q4</td>
<td>1.00</td>
<td>0.987</td>
</tr>
<tr>
<td>Q4D4</td>
<td>1.00</td>
<td>0.982</td>
</tr>
<tr>
<td>QM6</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>QM6D4-1</td>
<td>1.00</td>
<td>0.995</td>
</tr>
<tr>
<td>QM6D4-2</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>QM6D6</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

226–230, and references therein). All the problems are solved on a CONVEX-C1 in double precision (64 bits per floating-point word).

7.1. Plane stress patch tests

The first numerical simulation consists of patch tests for a plane stress problem, originally proposed in [26]. The uniform mesh configuration and skewed mesh configuration used in the test are shown in Fig. 4 and are consistent with [26]. The skewed mesh was originally devised to show the superior performance of QM6 to an earlier incompatible modes element in distorted mesh configurations.

The elastic coefficients are $E = 1$ and $v = 0.25$ for both cases. The first test case is an axial stretch with nodal loads equivalent to a pure axial normal stress of unit intensity applied along edge AB. The exact solution is $u_1 = x_1$ and $u_2 = -x_2/4$. (The origin of the two meshes is taken to be at point C.) The second test case is a linear varying normal stress of magnitude $x_2$ on edge AB. Due to antisymmetry of the problem, half of the mesh is modeled, with antisymmetry boundary conditions imposed on the nodes along edge AC. The bending solution to this problem $u_1 = x_1 x_2$ and $u_2 = -\frac{1}{2} (x_1^2 - v(x_2)^2)$.

Table 1 shows normalized results for the axial and bending deflections for the two mesh configurations evaluated at node A (Fig. 4). The axial stretch in both the configurations is exactly satisfied. The bending solution to an applied moment in the form of a couple is a quadratic polynomial. In the uniform mesh configuration, all elements with incompatible modes, with the exception of DM6D4-1, capture the solution exactly; this is because the incompatible modes contain quadratic terms in the coordinates of the parent domain $\xi$ and $\eta$, see (6.2) and (6.3), which coincide with the global coordinates for the uniform mesh and can therefore fully represent the exact solution. However, the incompatible modes are not complete second-order polynomials and, therefore, in the skewed mesh configuration, are unable to fully capture the exact displacement.

The axial stress at points a–g is exact for the axial stress case on both the mesh configurations. Therefore these results are not presented in tabular form. For the bending load case, the stresses obtained at points a–g are presented in Tables 2 and 3. As shown in Tables 1–3, the response of all the drill elements is slightly stiffer than the underlying displacement elements, confirming theoretical predictions. As for the displacements, on the uniform mesh configuration, all incompatible modes elements, with the exception of DM6D4-1, satisfy this test identically. Once again, for the distorted mesh configuration, the stresses are not exactly represented by any of these elements, since none of these elements contain complete biquadratic polynomials. However, from an engineering point of view, the improvement in accuracy over the Q4 and Q4D4 elements is substantial.

Table 2. Normalized axial stresses for the bending load case on the uniform mesh (except for point a, where the exact solution is zero)

<table>
<thead>
<tr>
<th>Element</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4</td>
<td>-0.033</td>
<td>0.988</td>
<td>0.988</td>
<td>0.987</td>
<td>0.986</td>
<td>0.987</td>
<td>0.998</td>
</tr>
<tr>
<td>Q4D4</td>
<td>-0.033</td>
<td>0.982</td>
<td>0.984</td>
<td>0.982</td>
<td>0.982</td>
<td>0.982</td>
<td>0.982</td>
</tr>
<tr>
<td>QM6</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>QM6D4-1</td>
<td>0.002</td>
<td>0.996</td>
<td>0.996</td>
<td>0.995</td>
<td>0.995</td>
<td>0.994</td>
<td>0.994</td>
</tr>
<tr>
<td>QM6D4-2</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>QM6D6</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 3. Normalized axial stresses for the bending load case on the skewed mesh (except for point a, where the exact solution is zero)

<table>
<thead>
<tr>
<th>Element</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q4</td>
<td>-0.008</td>
<td>0.798</td>
<td>1.204</td>
<td>0.957</td>
<td>0.908</td>
<td>0.986</td>
<td>0.946</td>
</tr>
<tr>
<td>Q4D4</td>
<td>-0.035</td>
<td>0.748</td>
<td>0.748</td>
<td>0.926</td>
<td>0.962</td>
<td>0.962</td>
<td>0.913</td>
</tr>
<tr>
<td>QM6</td>
<td>-0.007</td>
<td>0.910</td>
<td>0.908</td>
<td>0.986</td>
<td>1.001</td>
<td>0.999</td>
<td>1.003</td>
</tr>
<tr>
<td>QM6D4-1</td>
<td>0.002</td>
<td>0.882</td>
<td>0.882</td>
<td>0.983</td>
<td>0.967</td>
<td>0.967</td>
<td>0.994</td>
</tr>
<tr>
<td>QM6D4-2</td>
<td>0.002</td>
<td>0.908</td>
<td>0.908</td>
<td>0.985</td>
<td>1.000</td>
<td>1.000</td>
<td>0.999</td>
</tr>
<tr>
<td>QM6D6</td>
<td>0.002</td>
<td>0.908</td>
<td>0.908</td>
<td>0.985</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
</tr>
</tbody>
</table>
The elements based on the present formulation pass the patch tests whenever the underlying displacement elements pass these tests, in accordance with remark 4 in Section 5.

7.2. Single element tests

In this section we present various single element tests which serve as standard benchmark problems in element evaluation [24, 26]. Single element testing is an important ingredient in element development, investigating completeness of the interpolation polynomial and checking the constant stress states of the element. We looked at the constant stress states of some of these elements in the preceding subsection, and we will also study the effects of the completeness of the shape functions in Section 7.5. By studying a single element we can detect the rigid body modes, any false zero energy modes and invariance of the element under change in global orientation. Preliminary tests, not presented herein, indicate that our elements do not possess false zero energy modes and are invariant under change in global orientation. In the following tests we will look first at the performance of the quadratic elements and then linear elements.

7.2.1. Nine-node and six-node elements. Simple meshes of one nine-node element or two six-node elements (referring to the number of displacement nodes) for cantilever beam problems are subjected to bending moment and edge shear loadings as shown in Fig. 5 (see [26] for a detailed description of the configuration). For the edge shear test, we apply an equal but opposite nodal force at the roller support, as shown in Fig. 5. Elements involved in the test had both equal-order and unequal-order rotation fields (see Section 6.1). Thus, Q9-drill elements with equal-order and unequal-order rotation fields correspond to Q9D9 and Q9D4, respectively (Fig. 2, likewise for T6-drill elements). The normalized stresses and displacements at points a and b, respectively, obtained for the two load cases are presented in Tables 4 and 5. The exact solutions to an applied bending moment and an applied shear force are second-order and third-order polynomials, respectively. As can be seen from these tables, the quadratic elements capture the bending solutions exactly, whereas this is not the case for the shear force test. Numerical examples in the following sections show that with successive mesh refinements, the finite element solution converges to the exact solution at the optimal rate of convergence given by theory.

7.2.2. Four-node elements. We next consider the single element cantilever beam problem for testing the four-node elements presented in Section 6. The two load case are shown in Fig. 5. The normalized solutions for stresses at point a and displacement at point b (see Fig. 5) are given in Table 6.

As shown in the Table 6, the response of the standard bilinear quadrilaterals can be greatly improved by the incompatible modes technique, approaching the performance of quadratic elements. The addition of drill rotations has no effect on the solutions to these tests, since \( \Delta K = 0 \) for single elements, as predicted in remark 4 of Section 5.
7.3. Rate of convergence study

The configuration considered here is shown in Fig. 6 (see [34], pp. 219–220, 254–255). The problem is designed to assess the performance of plane stress/strain elements subjected to dominant in-plane bending behavior. In this study, plane stress conditions are assumed in force. A cantilever beam of length-to-depth ratio equal to four is subjected to a parabolically varying end load. Boundary conditions are set in accordance with an exact elasticity solution. Hence, nodal rotations are not constrained at the root. The exact solution depends on Poisson’s ratio; the value 0.3 is employed in the calculations presented herein. The meshes shown consist of 16 quadrilateral and 32 triangular elements. Finer meshes are constructed by bisection. In the case of the quadratic elements, a coarser mesh is also employed with one layer of elements through the depth.

Figures 7–9 present the numerical rates of convergence of these elements, measured in integral norms, which corroborate the theoretical predictions (4.13). As expected, no gain in the convergence rate for the quadratic elements is obtained by increasing the degree of the rotation interpolation from one to two.

Fig. 7. Convergence study for quadrilaterals.

Fig. 8. Convergence study for triangles (see key, Fig. 7).
(although equal-order elements do possess practical advantages). The results for linear triangles (Fig. 8) and bilinear quadrilaterals (Fig. 7) are somewhat better than anticipated. In general, one cannot expect better than first-order convergence for these elements in the norms considered. The line plots for the linear

![Graph showing convergence study for incompatible modes quadrilaterals.](image)

**Key:**
- $\|u^h - u\|_V$
- $\|\psi^h - \psi\|_W$

Fig. 9. Convergence study for incompatible modes quadrilaterals

![Graph showing convergence of tip deflection at the midfiber.](image)

**Key:**
- quadrilaterals
- triangles

- $k = l = 1$
- $k = 2, l = 1$
- $k = l = 2$
- QM6D4-2
- QM6D4-1

Incomp. modes

Fig. 10. Convergence of tip deflection at the midfiber.
Numerical assessment of membrane elements

triangles and the bilinear quadrilaterals are in fact a least-squares fit of the data points shown on the plots, and as we refine the meshes, an asymptotic value of the rate of convergence equal to one is obtained.

Figure 9 shows the convergence rates for the incompatible modes elements. The rates of convergence for displacements and rotations for QM6D6 are identical to that of QM6D4-2 and are therefore omitted from the plots. As expected, despite high accuracy on coarse meshes, no improvement over standard linear elements in asymptotic rate of convergence is obtained. (See Figs 7 and 9, and discussion in [36], pp. 174–181.)

7.4. Accuracy study

Accuracy of displacements at the midfiber, and axial stress at point A (Fig. 6) under the parabolically varying end load are presented in Figs 10 and 11, respectively. These plots not only show the pointwise convergence of displacements and stresses, but also the dramatic increase in coarse-mesh accuracy by increasing the order of interpolation. The accuracy of all the quadratic elements is excellent. As expected, the linear triangle and bilinear quadrilateral are comparatively stiff. Drilling elements with incompatible modes show accuracy comparable to the quadratic elements. Again, the results for QM6D6 are identical to QM6D4-2 and are therefore omitted from the plots.

7.5. Sensitivity to mesh distortion

This is a standard test for evaluating the sensitivity of the elements to mesh distortion. A cantilever beam, modeled by two quadrilateral elements, is subjected to a bending moment in the form of a couple (see Fig. 12). The edge separating the two elements is then
gradually rotated about its center, a distance of \( \pm \alpha \) on the top and the bottom surfaces, to distort the mesh. The degree of geometric distortion of these elements is represented by the dimension \( \alpha \). The same test is performed on triangles by bisecting the quadrilaterals. The elements used in the test are Q9-drills, T6-drills and Q4-drills with incompatible modes. Linear triangles and bilinear quadrilaterals are not presented because of their inability to capture bending behavior, even in the undeformed configuration.

The exact solution is a quadratic polynomial, and hence quadratic elements show no deterioration with mesh distortion as long as the edges are kept straight (Table 7), as is the case for the underlying elements. However, this is no longer the case when elements with curved edges are considered. Figure 13 shows an example of a configuration in which one edge of each element is curved. The ratio of the length of the side \( l \) and the horizontal offset from the straight line \( \Lambda \) gives an indication of the curvature. Keeping the position of the midside node along the curved side fixed, a given change in \( \alpha \) specifies a change in \( l \) and \( \Lambda \). Accuracy now degrades with mesh distortion, since the quadratic interpolation functions are no longer quadratic polynomials in the physical domain. Figure 14 shows the normalized displacements obtained at node A for the T6 and the two T6-drill elements. The accuracy of all three elements is almost identical, but compared to the exact solution the loss in accuracy with mesh distortion is substantial. Therefore, the extreme curvature of the sides of triangular elements is inadvisable.

The beam is also modeled with two drill elements with incompatible modes. The results are presented in Fig. 15. QM6D4-1 shows a stiff response, even without distortion, because the skew-symmetric part of the displacement field does not possess the incompatible modes, thus bringing in the inherent stiffness of the bilinear element. The other drill elements with incompatible modes exactly capture the exact solution in the undeformed configuration. The accuracy degrades with an increase in geometric distortion because these elements do not have complete quadratic interpolation functions and therefore are no longer quadratic polynomials in the physical domain.

### Table 7. Normalized displacements at node A

<table>
<thead>
<tr>
<th>Distortion ( \alpha )</th>
<th>Two Q9-drill quadrilaterals</th>
<th>Four T6-drill triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Q9</td>
<td>Q9D4</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

#### 7.6. Tip deflection and tip rotation vs \( \gamma \)

The meshes shown in Fig. 6 were used to evaluate the sensitivity of the formulation to varying \( \gamma \). The results obtained for the parabolically varying end load are reported in Figs 16–19. The stability analysis provides an upper bound of \( \gamma < \mu \) and all the elements tested show excellent results up to this limiting value. The performance of QM6D6 in the tip deflection and tip rotation tests is identical to that of QM6D4-2, therefore results for this element have been omitted from Figs 16 and 17. QM6D4-2,
QM6D6 and all the quadratic elements show very good accuracy properties well beyond the limit point. Deterioration in accuracy of tip deflection and tip rotation of these elements commences at about $\gamma/\mu = 10^{10}$, due to the fact that the $\gamma$-contributions numerically dominate the contribution of the other terms. However, the point at which the deterioration commences is beyond the region of practical interest.
Nonetheless, we have to respect the stability analysis for the proper performance of T3D3, Q4D4 and QM6D4-1.

7.7. Cook's membrane

This last problem, first proposed in [39] as a test case for general quadrilateral elements, shows the bending performance of the elements under excessive mesh distortion. The configuration is a tapered panel with one edge fixed and the opposite edge acted upon by a distributed shear load (see Fig. 20). Finer meshes are constructed by bisection. There is no known analytic solution for this problem but the results for a $32 \times 32$ mesh are used for comparison purposes. The vertical deflection at node C and the principal stresses at points A and B are shown in Figs 21–23, respectively.

Once again, the increased accuracy for the coarse mesh by increasing the order of the elements can be clearly observed. All quadratic elements show very good accuracy. All incompatible modes elements are superior to the parent Q4 element for this arbitrarily skewed mesh. The performance of QM6D6 is identical to that of QM6D4-2, and therefore has not been shown on these plots.

---

**Key:**

- quadrilaterals
- triangles

- $k = 2, l = 1$
- $k = l = 2$

---

Fig. 17. Tip rotation as function of $\gamma$ for linear elements (see key, Fig. 16).

Fig. 18. Tip deflection of $\gamma$ for quadratic elements.
8. CONCLUSIONS

The structural mechanics literature is replete with drill elements developed on intuitive geometrical concepts of in-plane rotations. Including the rotational degrees of freedom in membrane finite elements by using such techniques has not always proven successful in the derivation of invariant and convergent elements of arbitrary geometry and order. Engineering accuracy has been the driving force behind all such efforts, and elements based upon a mathematically sound variational formulation with general convergence proofs have so far been lacking.

In this paper we evaluated membrane finite elements with drilling degrees of freedom, based on a variational formulation that was devised from a simple, yet mathematically sound, theory. We view this work as the implementation and validation of a successful effort in bridging the gap between the mathematical and engineering communities; a link that seemed to be missing so far. Our experience shows that any standard element may be generalized to an element with drilling degrees of freedom by employing the technique presented herein, maintaining its accuracy and convergence properties. This generality is demonstrated by the development of a family of triangular elements, a family of Lagrange quadrilaterals and elements with incompatible modes. Extensive numerical testing confirms theoretical predictions and highlights the usefulness of these elements in engineering analysis. Specifically, the accuracy of the underlying element is essentially unaffected by incorporating drill degrees of freedom. Incompatible modes may be employed in the framework of our formulation to improve the coarse-mesh accuracy of bilinear elements in modeling bending behavior. Numerical studies show that the sensitivity of our elements to mesh distortion is comparable to that of the underlying elements. The parameter \( y \) may be varied over a wide range of values, in some cases well beyond the upper bound predicted by the theory, without affecting the numerical results. We recommend using a value of \( y = \mu \) for isotropic elasticity. Numerical experiments confirmed theoretical error estimates.

As predicted by the theory, quadratic elements with equal- and unequal-order rotation interpolation show the same rate of convergence, and numerical results indicate essentially identical accuracy as well. Elements with lower-order rotation fields have less degrees of freedom, and hence require somewhat less computational effort than elements with equal-order rotations, for results of the same quality. In contrast,
equal-order elements are more convenient from an implementational point of view, and the added computational effort required is not significant. Amongst the drill elements, we advocate the use of the QM6D4-2 element. This element shows superior performance to QM6D4-1, at the same cost, and shows identical performance to QM6D6, at a lower cost.

We have numerically validated the variational formulation presented herein as a general basis for the development of drill elements that retain the accuracy and convergence properties of the underlying elements. These ideas may be implemented in the development of specific elements. We are also examining the generalization of this method to dynamics and frequency analysis. For initiatory thoughts on nonlinear variations, see [40].
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Acknowledgement—We wish to thank R. L. Taylor for helpful comments.

REFERENCES


