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A SPACE-TIME FINITE ELEMENT FORMULATION FOR FLUID-STRUCTURE INTERACTION

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We present a space-time finite element formulation of the Navier-Stokes equations which is stabilized via the Galerkin/least-squares approach. The variational equation is based on the time discontinuous Galerkin method and is written in terms of the physical entropy variables over the moving and deforming space-time slabs. The formulation thus becomes analogous to the classical arbitrary Lagrangian-Eulerian technique and is therefore applicable to a broad spectrum of flow problems that involve moving free surfaces and deforming fluid-solid interfaces. Numerical simulation of a projectile moving in a stationary flow field is presented to show the versatility of the proposed methodology.

1. INTRODUCTION

Fluid flow problems that involve free surfaces and moving fluid-solid interfaces add a level of complexity to flows over fixed spatial domains [1,11,15]. Such problems have classically been modeled via Lagrangian as well as quasi-Eulerian techniques [2]. However, both approaches are seriously limited in their scope for such complicated flow fields. In order to deal with flows involving deforming spatial domains, the so-called arbitrary Lagrangian-Eulerian finite element techniques are usually employed (for a review see [2,4,9] and references therein). In this paper we have adopted a different approach, which is based on the idea of space-time Galerkin/least-squares finite element calculations of the Navier-Stokes equations over deforming configurations [12-14,17,18]. The basis of this formulation is a time-discontinuous Galerkin method [7,16]. The conceptual framework of the underlying Galerkin/least-squares method has evolved as a generalization of the streamline-upwind/Petrov-Galerkin method (SUPG), developed earlier by Hughes and Brooks [3] for convective transport problems.

A summary of the paper is as follows: The Navier-Stokes equations are introduced in Section 2. Strong form of the initial/boundary value problem is presented in Section 3. Section 4 presents the space-time finite element description, which, in conjunction with appropriate restrictions on the mesh motion can lead to Lagrangian, Eulerian or arbitrary Lagrangian-Eulerian frameworks. The weighted residual formulation stabilized via the Galerkin/least-squares approach is presented next. Sample numerical results are presented in Section 6 and conclusions are drawn in Section 7.

2. THE NAVIER-STOKES EQUATIONS

In terms of conservation variables, the compressible Navier-Stokes equations can be written as

$$\mathbf{U}_t + \mathbf{F}_x = \mathbf{F}_s + \mathbf{F}$$

where $\mathbf{U}$ is the vector of conservation variables, $\mathbf{F}_s$ and $\mathbf{F}_f$ are respectively the vectors of advective and diffusive fluxes in the $i$th direction and $\mathbf{F}$ is the source vector. See [8,12] for details.

We add the following constitutive relations to the conservation equations for mass, momentum and energy.

$$\tau_{ij} = \lambda \ u_{k,k} \ \delta_{ij} + \mu \ (u_{i,j} + u_{j,i})$$

$$\kappa = -\kappa T_i$$

where $\lambda$ and $\mu$ are the viscosity coefficients and $\kappa$ is the coefficient of thermal conductivity. Equation (2) defines the viscous stress components and (3) is Fourier's law of heat conduction.

Remark: A linear equation of state for a slightly compressible fluid under isothermal conditions can be written by expanding $p$ in terms of $\rho$

$$p - p_{ref} = \frac{\beta}{\rho_{ref}} (\rho - \rho_{ref})$$
where $\beta$ is the bulk-modulus, while $p_{\text{ref}}$ and $\rho_{\text{ref}}$ are constant reference values of pressure and density.

We can write the system of equations in a quasi-linear form giving rise to a non-symmetric system of partial differential equations. In terms of entropy variables, equation (1) is symmetrized via a change of variables by Hughes et al. [8]. See [8] for explicit definition of the flux vectors and coefficient matrices.

3. STRONG FORM OF THE INITIAL / BOUNDARY VALUE PROBLEM

Let $\Omega_t$ be an open, bounded domain in $\mathbb{R}^n$, where $n_{sd}$ is the number of space dimensions. The closure of $\Omega_t$ is $\bar{\Omega}_t$ and the boundary of $\Omega_t$ is denoted by $\Gamma_t$. It is important to note that in the present case the geometry of the spatial domain and the spatial boundary are also time-dependent. The unit outward normal vector to $\Gamma_t$ is denoted by $\mathbf{n} = \{n_i\}$.

We also assume that $\Gamma_t$ admits the following decomposition:

$$\Gamma_{g_t} \cup \Gamma_{h_t} = \Gamma_t$$  \hspace{1cm} (5)

and

$$\Gamma_{g_t} \cap \Gamma_{h_t} = \emptyset$$  \hspace{1cm} (6)

where $\Gamma_{g_t}$ and $\Gamma_{h_t}$ decompose the boundary $\Gamma_t$ into regions where essential and natural boundary conditions, respectively, are prescribed [6]. All data are assumed to be functions of space ($x$) and time ($t \in [0, T]$).

Let $\mathcal{L}$ be the differential operator for the Navier-Stokes equations

$$\mathcal{L} = \tilde{\mathbf{A}}_0 \frac{\partial}{\partial t} + \tilde{\mathbf{A}}_1 \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} (\tilde{\mathbf{K}}_{ij} \frac{\partial}{\partial x_j})$$  \hspace{1cm} (7)

The formal statement of the initial/boundary-value problem under consideration is: Given $\mathbf{V}_0 : \Omega_0 \rightarrow \mathbb{R}^{n_{sd}}$, $g : \Gamma_3 \times [0, T] \rightarrow \mathbb{R}^{n_{sd}}$ and $h : \Gamma_h \times [0, T] \rightarrow \mathbb{R}^{n_{sd}}$, find $\mathbf{V} : \Omega \times [0, T] \rightarrow \mathbb{R}^{n_{sd}}$ such that for all $\mathbf{W} \in \mathbf{S}_{n_{sd}}$, $t \in [0, T]$

$$\int_{\Omega} (\mathbf{L}^\top \mathbf{W}) \cdot \mathbf{dV} = 0$$  \hspace{1cm} (8)

$$\mathbf{V}(x, 0) = \mathbf{V}_0(x) \quad \forall x \in \Omega_0$$  \hspace{1cm} (9)

$$\mathbf{q}(\mathbf{V}) = g \quad \text{on} \quad \Gamma_g \times [0, T]$$  \hspace{1cm} (10)

$$(-\mathbf{F}_1 + \mathbf{F}_2)_{i_1} = h \quad \text{on} \quad \Gamma_h \times [0, T]$$  \hspace{1cm} (11)

where $n_{\text{doof}} = n_{sd} + 2$ is the number of degrees of freedom, $\mathbf{q}$ is a nonlinear boundary condition function, and $\mathbf{g}$ and $\mathbf{h}$ are the vectors of prescribed essential and natural boundary conditions, respectively.

Remark: In practice it is convenient to implement the boundary and initial conditions in terms of primitive variables.

4. A SPACE-TIME DESCRIPTION OF THE MOVING DOMAINS

Let $I = [0, T]$ be an open time interval partitioned into an ordered series of time levels with $\Omega_n$ and $\Gamma_n$ as the approximations to the $d$-dimensional spatial domain $\Omega$ with boundary $\Gamma$ at time level $t_n$. A space-time "slab" $Q_n$ in the context of moving domains is then defined as the region enclosed between $\Omega_n$, $\Omega_{n+1}$ and $P_n$ where $P_n$ is the surface described by the boundary $\Gamma_t$ as it traverses $I_n$. The displacement field of the spatial mesh is assumed continuous in time. Consequently a space-time slab deforms during a time-step while the initial configuration of any slab is identical to the final configuration of its previous slab.

Remark:

1. It can be proved that if we write the local form of a conservation law in the spatial space-time domain, going through the transformation between the deforming physical domain and the moving reference domain, we obtain the equations written in an arbitrary Lagrangian-Eulerian framework [12, 14].

2. Since the finite element functions are discontinuous at the space-time slab interfaces, the spatial discretization can be changed from one space-time slab to another, and thus provides a natural mechanism for incorporating adaptive remeshing [10] in the formulation.

5. GALERKIN/LEAST SQUARES WEIGHTED RESIDUAL FORMULATION

The formal statement is that within each space-time slab $Q_n$, $n = 0, \ldots, N - 1$, find $\mathbf{V}^h \in S^h_n$ such that for all $\mathbf{W}^h \in V^h_n$ the following variational equation is satisfied

$$\int_{Q_n} \left(-\mathbf{W}^h : \mathbf{U}(\mathbf{V}) - \mathbf{W}_3^h \cdot \mathbf{F}_1(\mathbf{V})ight) \cdot \mathbf{dV}$$

$$+ \mathbf{W}_3^h \cdot \tilde{\mathbf{K}}_{ij} \mathbf{V}^h_j - \mathbf{W}^h \cdot \tilde{\mathbf{F}} \right) dQ$$

$$+ \int_{\Omega_{n+1}} \left(\mathbf{W}^h \cdot \mathbf{F}_i(\mathbf{V}^h) \right) \cdot \mathbf{U}(\mathbf{V})(\mathbf{V}^h_i) \right) d\Omega$$

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\[- \int_{Q_n} \left( W^h(t_n^+) \cdot U(V^h(t_n^+)) \right) d\Omega \]
\[+ \sum_{e=1}^{(n_{el})_m} \int_{Q_n^e} \left( L W^h \right) \cdot \left( \mathbf{L} V^h - \mathbf{F} \right) dQ \]
\[+ \sum_{e=1}^{(n_{el})_m} \int_{Q_n^e} \nu \delta_{e} \mathbf{V}^h \cdot \mathbf{A}_0 \delta \mathbf{V}^h dQ \]
\[= \int_{P_n} \mathbf{W}^h \cdot \left( - \mathbf{F}_f(V^h) + \mathbf{F}_f(V^h) \right) n_i dP \quad (12)\]

The first three integrals on the left-hand side, together with the last integral on the right-hand side, constitute the time-discontinuous Galerkin formulation. Integration-by-parts of the time flux term gives rise to the time boundary integral which is added to the jump condition to give the second and the third integrals. These terms provide a mechanism by which information is propagated from one space-time slab to the next.

The fourth integral in (12) is the least-squares operator, which is only defined on element interiors. \( L \) is the differential operator defined in (7), \( \mathbf{L} V^h - \mathbf{F} \) is the residual of the Navier-Stokes equations, and \( \tau \) is a \( n_{\text{dof}} \times n_{\text{dof}} \) symmetric positive semidefinite matrix of intrinsic time scales. Its presence in the formulation enhances the stability of the numerical solution.

The fifth integral in (12) is the discontinuity capturing operator. The computation of flow fields that involve shock waves requires the inclusion of such an operator to accurately model the jump in the field variables across the shock waves. This is a residual based operator and captures the sharp gradients in the solution. It acts in the direction of the gradients and controls the oscillations in the solution while maintaining higher order accuracy by vanishing quickly in the smooth regions of the solution. For details see [16].

The boundary integrals on the right-hand side of (12) give rise to natural boundary conditions. This space-time weighted residual formulation is consistent since the exact solution of the Navier-Stokes equations, \( \mathbf{V} \), also satisfies the variational equation.

Remark:
1. The weighted residual formulation is written on the current domain and is stabilized via a least-squares integral [7] which is only defined in the interior of the elements, thus preserving the consistency of the Galerkin method.

2. In the Galerkin/least-squares framework, general combinations of the interpolation functions, which otherwise violate the critical stability conditions in the Galerkin framework, also become convergent.

6. NUMERICAL RESULTS

The numerical simulation presented here is that of a projectile moving in a stationary flow-field at Mach 0.01 and Reynolds number 1000. The Reynolds number is measured using the mean flow data and the velocity and length of the projectile. The boundary conditions for this problem are shown in Fig. 1. The computational domain for this problem covers an area \(-12 < x < 12 \) and \(-20 < y < 20 \), with a unit length projectile having its tail placed at \( x = 0 \) and \( y = 0 \). To solve this problem, an unstructured triangular mesh comprising 7408 linear triangles with 3844 nodes was generated. We used a mesh rezoning strategy [12,14] to move the projectile through the fluid domain.

![Fig. 1. Schematic diagram of the problem.](image)
the projectile are presented in the figures 3-6. We are looking through a window fixed in space and watch the projectile fly by. The spatial mesh moves and thus deforms at every time step and requires recomputation of the geometric parameters of the space-time elements (shape-function derivatives, Jacobians, etc.). We have plotted the pressure variations above and below the mean background pressure. A high pressure region develops at the tip and a low pressure region develops at the tail of the projectile. This variation in pressure causes the fluid to flow towards the tail end and can be seen in the velocity vector plots. We can also observe a well defined wake which is being dragged along with the projectile. This pattern is also confirmed in the streamline plots presented below.

The dependent variables of practical importance are the drag coefficient \( CD = \int_0^\infty \sigma_{2n} d\phi / \pi \rho_\infty u_p^2 \) and the side-sway coefficient \( C_w = \int_0^\infty \sigma_{1n} d\phi / \pi \rho_\infty u_p^2 \)
where \( \sigma_{2n} \) and \( \sigma_{1n} \) are the components of the normal stress \( \sigma_n \), \( \rho_\infty \) is the free stream density and \( u_p \) is the velocity of the projectile. Figure 2 shows the temporal development of drag and side-sway coefficients. We see some oscillation up to the tenth step, which is due to the start-up from rest. However, once the projectile attains a uniform velocity, this oscillation disappears.

![Fig. 2. Drag and side-sway forces.](image)

To solve this problem we used an implicit, third-order predictor multi-corrector algorithm with five corrector passes. The maximum dimension of the Krylov space was 10. A global time stepping strategy with time increment \( \Delta t = 0.01 \) was used. The total CPU time for these computations was 4.61 hours of which 6.6% time was used in mesh rezonings.

7. CONCLUDING REMARKS

We have presented a stabilized space-time finite element formulation of the Navier-Stokes equations which is written on the spatial space-time domains and allows equal-order interpolation for velocity and pressure. Kinematically, this approach is analogous to the classical arbitrary Lagrangian-Eulerian technique. However, the added advantage in the proposed approach is that by using the time-discontinuous Galerkin method, we can have an entirely new spatial discretization from one space-time slab to another. This is particularly important in situations where the mesh gets exceedingly distorted because of the movement of the computational domain and can potentially induce numerical errors. The present approach allows us to naturally accommodate adaptive refinements in the solution. The method has been tested for viscous and inviscid flow problems and shows excellent agreement with the experimental results.

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9. REFERENCES


Fig. 3. Mesh and flow field around the projectile at step 25.
Fig. 4. Mesh and flow field around the projectile at step 50.
Fig. 5. Mesh and flow field around the projectile at step 75.
Fig. 6. Mesh and flow field around the projectile at step 100.